NZEST
NEW ZEALAND
EDUCATION & SCHOLARSHIP TRUST
ACADEMIC ACHIEVEMENT BY CHOICE

NZEST SCHOLARSHIP EXAMINATION

1994 EXAMINER'S REPORT AND SOLUTIONS

MATHEMATICS WITH CALCULUS

NZEST Mathematics with Calculus Examiner's Report, 1994

This year's paper has continued the now well-established pattern, while incorporating several of the excellent suggestions teachers have made for improvements. So we continue to have six 20-mark questions of contrasting types. Most involve various sorts of written problems or unexpected applications. This year question one was merely a survey question of basic techniques, as some candidates had been put off by meeting unfamiliar wordy problems on the first page. Throughout the paper an attempt has been made to reduce the amount of words, which many saw as excessive in 1993. There are still sizeable paragraphs. I do not apologise for this, as it is the way innovative New Zealand teachers have been saying we should be going for years. There has been an attempt to grade the level of difficulty. Not every question in a scholarship paper needs to be profoundly difficult. Hence questions 1 to 4 are searching, but not aggressively difficult, while questions 5 and 6 are significantly harder.

Exam Statistics :

Number sat :	701									
Median :	57 %									
Upper quartile :	68 %									
Lower quartile :	44 %									
Percentage :	0-	10-	20-	30-	40-	50-	60-	70-	80-	90-
Frequency :	3	17	46	65	124	137	154	81	55	18

The over-all standard is clearly very good, and a credit to all the effort staff and students have put into preparation. As usual the scripts were a pleasure to mark, containing many original and neat ideas. Numerous cheery comments indicated students who were enjoying themselves. Almost all students got their actual raw mark (out of a possible 120) as a percentage. Only five scored over 100. Only 10 (with raw totals over 95) needed to be adjusted slightly to map into the final percentage. In the time given I consider it reasonable to expect a student to complete about five questions. Because of the problem-solving content, I think it is best to allow total freedom, and encourage the student to have a go at any or all of the questions. Hesitancy at getting started (often pinned to the terrible question *"Is this the best question for me to choose ?"*) is one of the main factors that contribute to poor performance in problem-solving questions.
A new feature has been used in the marking scheme. About six marks in each question are coded with the letter *E*. The marker is here encouraged to use plenty of latitude and award the mark if any sensible *starting activity* is presented. A further six marks are coded *H*. These mostly relate t things one is asked to prove. Only technically correct mathematics will do here. The intention is to encourage students to have a go. Especially in the initial stages of each problem, any reasonable response is likely to get some marks. Then, once the student has got warmed up to the question, we begin to look for a higher level of work.
Having now reached the end of my three-year term, I would like to thank mathematics staff throughout New Zealand, and especially those on the marking panel, for supportive and constructive feed-back. The high standard of candidate being presented indicates excellent seventh form teaching, with a commitment to fostering virile innovative approaches rather than the canned repetitive technical methods of yesteryear.

Question 1

My instructions to the markers :
This is a straight-foward survey of some basic techniques. Watch for any alternative (but valid) methods. Penalize mistakes, but not glosses. I hope this question will give a median of 12.

How the students got on :

Median :	12
Upper quartile :	15
Lower quartile :	9
Number scoring 20 :	17

What the markers thought :

(a) was well-done. About one-third got (b) wrong, often due to difficulties with the limits of integration, or by assuming F(0) is always zero. (c) was done well, with almost all who separated the variables correctly, completing it. (d), the hard part, was often omitted, or done poorly.

The marking scheme :

(a) $e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$

$e^{-x} = 1 - x + x^2/2! - x^3/3! + x^4/4! - \ldots$ [E]

$e^x - e^{-x} = 2x + 2x^3/3! + \ldots$ [E]

$[e^x - e^{-x}]/2 = x + x^3/3! + \ldots$ [convergent for all $x \in R$] [√]

Similarly

$[e^x + e^{-x}]/2 = 1 + x^2/2! + \ldots$ [convergent for all $x \in R$] [√]

$\text{Sinh}(x) = [e^x - e^{-x}]/2$

$\text{Sinh}'(x) = [e^x - (-e^{-x})]/2$ [E]

$= [e^x + e^{-x}]/2$

$= \cosh(x).$ [√]

$\text{Cosh}'(x) = \sinh(x)$ [E]

(b) Let $u = 2x+5$

So $x = (u-5)/2$

$dx/du = 1/2$ [E]

$$\int_0^5 (2x-5)\sqrt{2x+5}\,dx = \int_5^{15} (u-10)u^{0.5}\frac{dx}{du}\,du$$ [H]

$$= \int_5^{15} \left(u^{1.5} - 10\,u^{0.5}\right) 0.5\,du$$

$$= \left[\frac{1}{5}u^{2.5} - \frac{10}{3}u^{1.5}\right]_5^{15}$$

$= 6.7225$ [√]

$$\int_0^5 (2x+5)\sqrt{2x-5}\,dx$$

cannot be evaluated because the function $\sqrt{(2x-5)}$ is not defined for all the interval [0,5]. [E]

(c)

$$\int \frac{dy}{y+1} = \int dx$$ [√]

$\ln(y+1) = x + C$ [√]

$y+1 = e^{x+C}$

$y+1 = k.e^x$ (where $k = e^C$)

$y = k.e^x - 1 .$ [√]

Substituting $x=0, y=4$:

$4 = k - 1$

$k = 5$

So $y = 5.e^x - 1 .$ [√]

(d)

To prove : $\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \ldots + \binom{5+n}{n} = \binom{6+n}{n}$ . ... *

<u>Step 1</u> : Prove * is true when n = 1

$$\text{LHS} = \binom{5}{0} + \binom{6}{1} = 1 + 6 = 7$$

$$\text{RHS} = \binom{7}{1} = 7 = \text{LHS}$$ [H]

Step 2 : Assume * is true when n = k.
That is

$$\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \ldots + \binom{5+k}{k} = \binom{6+k}{k}$$ ....# (for some $k \in N$) [H]

Step 3 : Prove that # implies * is also true when n = k+1.

Take # and add $\binom{6+k}{k+1}$ to both sides.

$$\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \ldots + \binom{5+k}{k} + \binom{6+k}{k+1} = \binom{6+k}{k} + \binom{6+k}{k+1}$$ [H]

$$= \binom{7+k}{k+1}$$ (by the given result), [H]

which is the same as * with n = k+1.

Step 4 :
Let k = 1, and thus * is true for n = 2.
Let k = 2, and thus * is true for n = 3. ... Thus * is true for all $n \in N$ by induction.
For a logical proof : [H]

Question 2

My instructions to the markers :
This looks at the idea of parametric equations as a unifying theme between Cartesian graphs and sets of complex numbers. The first parts are intended to be easy, with the student gradually led along to the formidable challenge of part (e). It is important that the student thinks carefully about what is going on during the first parts. Students who over-use plotting points should get some credit in each part, but certainly not full marks. I expect a median of 12.

How the students got on :
Median : 10
Upper quartile : 14
Lower quartile : 7
Number scoring 20 : 9

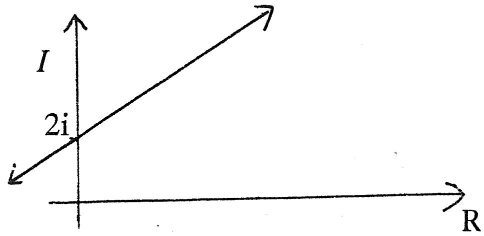
What the markers thought :
(a), (b) and (c) were well-done. A common error was to include i as part of *Im*(z). In (d) many answers were good, although those who chose to make t the subject usually had trouble taming a page of horrendous algebra. It is much easier to merely substitute into the given equation (see below). Although most found (e) impossible, about a dozen did solve it, employing a range of delightful techniques.

The marking scheme :

(a)

t :	0	1	2	3	4
z(t) :	1+3i	2+4i	3+5i	4+6i	5+7i

[EE]



[E]

(b) $x - y = -2$ [E]

$y = x + 2$ [or $x - y + 2 = 0$]. [E]

(c) $z(t) = (t+i)^2$

$= t^2 - 1 + 2t\,i$ [E]

Let $x = t^2 - 1$

$y = 2t$ [H]

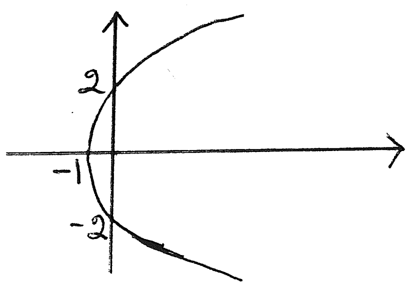
$y2 = 4t^2$

$t^2 = y^2/4$ [√]

$x = y^2/4 - 1$,

that is $y^2 = 4(x+1)$, [√]

which is the equation of this parabola :



[E]

(d) $z = \dfrac{t-i}{t+i}$

$= \dfrac{(t-i)^2}{(t+i)(t-i)}$

$= \dfrac{t^2 - 1 - 2t\,i}{t^2+1}$

$= \dfrac{t^2 - 1}{t^2+1} + \left(\dfrac{-2t}{t^2+1}\right)i$ [H]

Let $x = \dfrac{t^2 - 1}{t^2+1}$, and $y = \dfrac{-2t}{t^2+1}$. [√]

$x^2 = \dfrac{(t^2-1)^2}{(t^2+1)^2} = \dfrac{t^4 - 2t^2 + 1}{(t^2+1)^2}$ [√]

$y^2 = \dfrac{4t^2}{(t^2+1)^2}$

So, $x^2 + y^2 = \dfrac{t^4 - 2t^2 + 1 + 4t^2}{(t^2+1)^2}$ [√]

$= \dfrac{t^4 + 2t^2 + 1}{(t^2+1)^2}$

$= \dfrac{(t^2+1)^2}{(t^2+1)^2}$

$= 1.$ [√]

(e) $z = \dfrac{t+ki}{t+i}$

$= \dfrac{(t+ki)(t-i)}{(t+i)(t-i)}$

$$= \frac{t^2 + k + (kt-t)\,i}{t^2+1}$$

Let $x = \dfrac{t^2+k}{t^2+1}$, and $y = \dfrac{(k-1)t}{t^2+1}$. ....(*) [√]

$$x = 1 + \frac{k-1}{t^2+1}$$

$$y = (x-1)\,t$$

$$t = \frac{y}{x-1}.$$ [H]

Substitute back into equation *.

$$y = \frac{(k-1)\left(\frac{y}{x-1}\right)}{\left(\frac{y}{x-1}\right)^2 + 1}.$$

$$= \frac{(k-1)\,y}{(x-1)} \cdot \frac{(x-1)^2}{\left[y^2 + (x-1)^2\right]}$$

$$y\,(x-1)\left[y^2 + (x-1)^2\right] = (k-1)\,y\,(x-1)^2$$

$$y^2 + (x-1)^2 = (k-1)(x-1)\text{, provided } x \neq 1$$

$$y^2 + x^2 - 2x + 1 - (k-1)\,x + (k-1) = 0$$ [H]

$$y^2 + x^2 + (-k-1)\,x + k = 0$$

$$y^2 + (x - [k+1]/2)^2 + k - ([k+1]/2)^2 = 0$$

$$y^2 + (x - [k+1]/2)^2 + k - [k^2+2k+1]/4 = 0$$

$$y^2 + (x - [k+1]/2)^2 - [k^2-2k+1]/4 = 0$$

$$y^2 + (x - [k+1]/2)^2 = ([k-1]/2)^2$$ [H]

Which is the equation of a circle, with centre $([k+1]/2, 0)$ and radius $|[k-1]/2|$. [H]

Question 3

My instructions to the markers :

This is a practical application of numerical integration. All parts are intended to be straight forward. There is the possibility of a candidate getting bogged down unless they can keep a cool head and retain a sense of direction. Some may be unfamiliar with the idea of the numerical method for finding volume, so in part (c) they are given some assistance. Those who try to dodge the heavy arithmetic by rounding all the numbers will be penalised, but still get some credit. There will be mistakes concerning units. Subtract no more than a maximum of 1 mark off for this. I hope this will be a high-scoring question, and am picking a median of 14.

How the students got on :

Median :	10
Upper quartile :	13
Lower quartile :	6
Number scoring 20 :	7

What the markers thought :

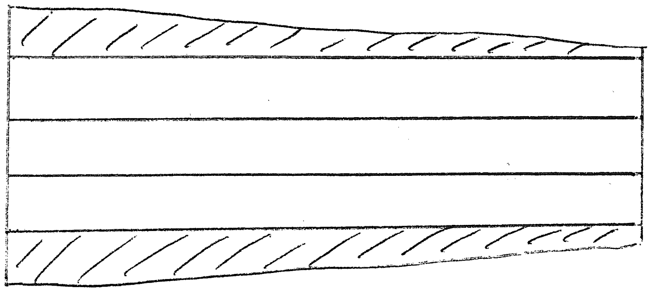
It was a pity many spoiled their work with confused thinking about units, often mixing mm and m in the same line of working. Another common mistake was to take h as the number of intervals, instead of as the size of each interval. In (b) almost all diagrams were far too rough. Many had very little spatial awareness. Infeasible diagrams were marked wrong. For (c) too many plotted a graph with diameters instead of areas on the y-axis. There were a lot of fine and ingenious answers to (d), although not all realised they needed to look at the cross-section 4 metres along the log. (e) was often done well. (f) was an easy mark, provided a sensible comparison was made.

The marking scheme :

(a) Area = $1000/2\,[\,610 + 2(620+610+610+590) + 580]$ [EE]

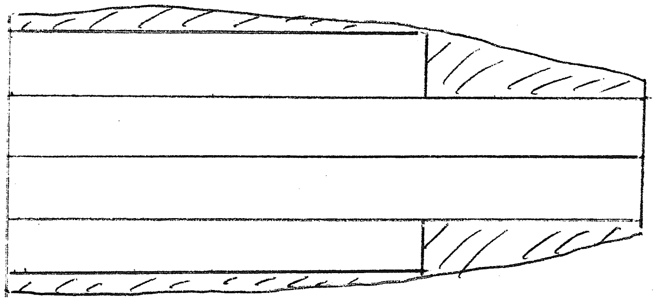
$= 3{,}025{,}000\ \text{mm}^2\quad [\,3.025\ \text{m}^2\,]$. [H]

(b)



[E]

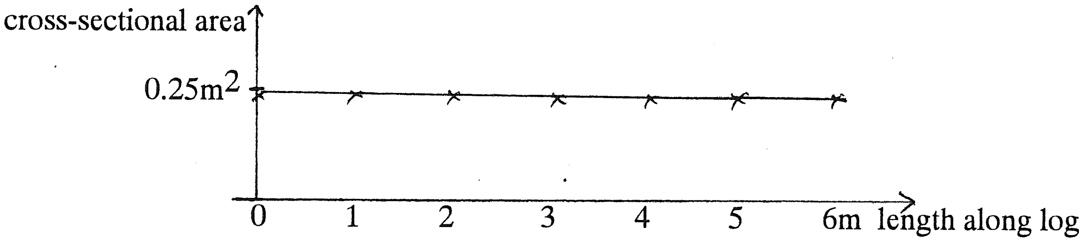
<u>Method 1</u> : cut 3 planks each 5m long giving 2.25 m^2 of saleable timber.



[E]

<u>Method 2</u> : cut 2 planks each 5m long and 2 planks 3m long giving 2.4 m^2 of saleable timber.
Thus 4 planks can be cut. [√]
Method 2 is better for reasons shown above. [√]

(c)

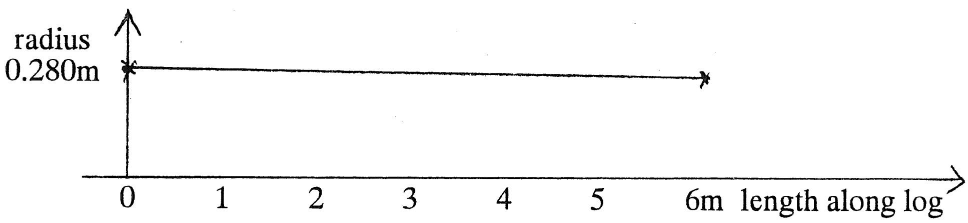


[E]

Volume = 1000/3[246301 + 4(237583+229022+212372) + 2(237583+220618)] [√√]
= 1363661000 mm^3 [1.363661 m^3]. [H]
Correct use of units in (c) : [√]

(d) If we are to cut the required beam, its cross-section must fit inside the log cross-section located at 4 metres from the thick end of the log. That is, it must fit inside a circle with diameter 530mm (using given assumption about symmetry). [√]
The cross-section of the beam is rectangular, with a diagonal of 540.8mm (Pythagoras). [√√]
Thus it is impossible. For logical reasoning : [H]

(e)



[√]

This line has equation $y = -0.003333x + 0.28$ (using metre units).
The area of a typical cross-section $= \pi\,(-0.003333x + 0.28)^2$
$= \pi\,(0.00001111x^2 - 0.0009333x + 0.0784)$ [√]

$$\text{volume} = \pi \int_0^6 (0.00001111x^2 - 0.0009333x + 0.0784)\ dx$$ [H]

$$= \pi \left[\left(0.00001111\frac{x^3}{3} - 0.0009333\frac{x^2}{2} + 0.0784x\right) \right]_0^6$$

$= 1.37476\ m^3$ [H]

(f) The method in part (c) is much more likely to be accurate, as it utilizes all the given measurements, and not just the two end measurements. [√]

Question 4

My instructions to the markers :
This is an investigative question. One has to look at four alternative ways of subdividing the land, and decide which of these is the best. I hope students find this interesting. The general idea should not be too unfamiliar to them. When marking, be very alert to any unexpected, but perhaps valid, interpretations. Let's try for a median of 12.

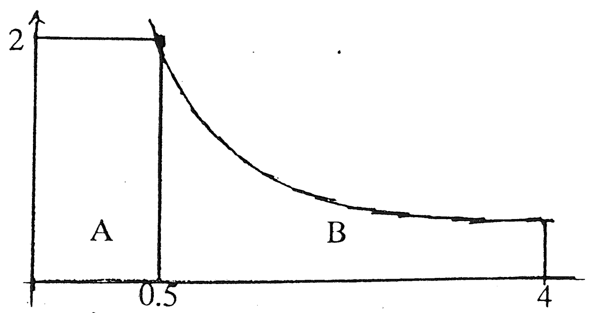
How the students got on :

Median :	15
Upper quartile :	17
Lower quartile :	11
Number scoring 20 :	27

What the markers thought :
Students seemed to like this question. Generous consistency (*follow-through*) marking was allowed. (a) and (b) were done well, although some chose the wrong limits of integration and had trouble with ln(0). Many lost easy marks by not drawing the diagrams asked for in (c). Answers to (d) which were impossibly large (bigger than 4km) were penalised. Part (e) gave the students scope to demonstrate a wide repertoire of clever methods of solution. Those who relied on intuition to come up with the point (1,1) only got 1 mark here.

The marking scheme :

(a)



Area A = 1 [E]

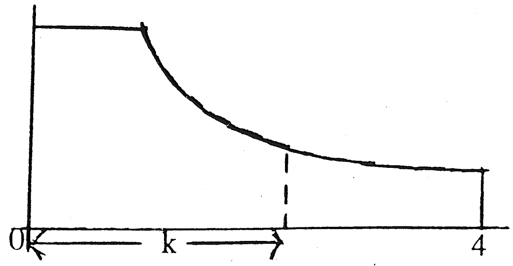
$$\text{Area B} = \int_{0.5}^{4} \frac{1}{x}\,dx$$ [√]

$$= [\ln x]_{0.5}^{4}$$

$$= \ln 8$$ [√]

Total area $= 1 + \ln 8\ \text{km}^2$. [= 3.07944 km^2]

(b) Required area for camp = 3.07944 / 4 = 0.76986 . [√]
Define k thus :

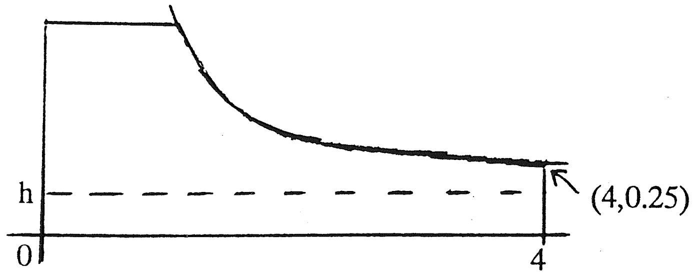


Thus we require :

$$\int_k^4 \frac{1}{x}\,dx = 0.76986$$ [E]

$$\begin{aligned} \ln 4 - \ln k &= 0.76986 && [\surd] \\ \ln k &= \ln 4 - 0.76986 \\ &= 0.616434 && [\surd] \\ k &= 1.852 \text{ km} && [\surd] \end{aligned}$$

(c) <u>method 1 :</u>

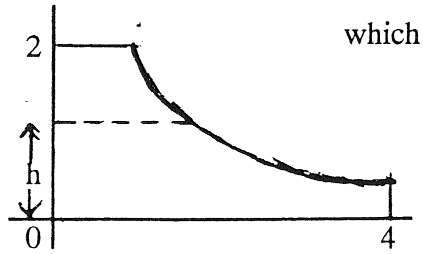


[E]

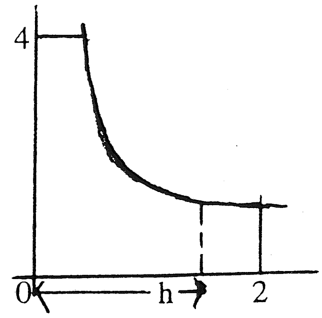
We require $4h = 0.76986$

$h = 0.192$ km (Observe this is < 0.25, so diagram is valid). [√]

<u>method 2</u> :



which reflects to :



[E]

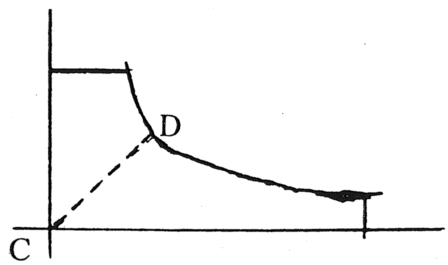
Thus we require :

$$\int_h^2 \frac{1}{x}\,dx = 0.76986$$ [H]

$$\begin{aligned} \ln 2 - \ln h &= 0.76986 \\ \ln h &= \ln 2 - 0.76986 \\ &= -0.0767128 \\ h &= 0.926156 \text{ km} && [\surd] \end{aligned}$$

(d) Length of new fence for (b) = 1/1.85231 = 0.54 km [E]
Length of fence for (c) method 1 = 4 km [√]
Length of fence for (c) method 2 = 1/0.926156 = 1.08 km [√]
So part (b) gives the shortest fence. [√]

(e)



The idea is to show that the minimum length of CD occurs when D = (1,1), ...*
giving a minimum length of $\sqrt{2}$ = 1.414 km, which is much worse than part (b) above. [H]

Proof of * : Let D = (x, 1/x)

$$\text{Let } f = CD^2 = x^2 + x^{-2}.$$
$$f' = 2x - 2x^{-3}$$

Let f' = 0 :

$$2x - 2x^{-3} = 0$$
$$2x^4 = 2$$
$$x = 1 \quad (x > 0 \text{ in application}).$$ [H]

$$f''(x) = 2 + 6x^{-4}$$

$f''(1) = 8 > 0$, so $x = 1$ at the minimum value of CD^2, and thus the minimum of CD. [H]

Question 5

My instructions to the markers :

This gives a commercial application which involves geometric sequences, geometric series, and binomial expansions. It is an algebraic question, but the main emphasis is on requiring accurate proofs. Most students will find this searching. Sloppy reasoning will be penalised. All necessary reasons must be given. I hope we can get a median of 10.

How the students got on :

Median :	5
Upper quartile :	8
Lower quartile :	2
Number scoring 20 :	1

What the markers thought :

Nearly all the answer to (a) was already provided in the question, but despite this generous start, most students experienced profound difficulties. Few realised the binomial theorem was needed in (b). The skill of proving something from the analysis of a series seems to be a lost art. Many could not simplify the factorials in (c). Although (d) was very hard, there were some courageous attempts to deal with it case by case and thus build up a pattern. Very few seemed to understand what (e) was asking, those answering correctly getting well-earned marks.

The marking scheme :

(a) Preliminary sketch eg :

After 1 year the amount is $A(1+ k/100)$.

After 2 years the amount is $A(1+ k/100)^2$.

After 3 years the amount is $A(1+ k/100)^3$. [E]

This is a geometric *sequence*, with a first term of $a = A(1+ k/100)$, [E]

and a common ratio of $r = (1+ k/100)$. [√]

Thus the n-th term will be $A(1+ k/100)^n$. [E]

(b) To prove : $A(1+ nk/100) \leq A(1+ k/100)^n$.

$$\Leftrightarrow \quad 1+ nk/100 \leq (1+ k/100)^n.$$ [√]

Expanding : $\text{RHS} = 1 + \dfrac{nk}{100} + \binom{n}{2}\dfrac{k^2}{100^2} + \binom{n}{3}\dfrac{k^3}{100^3} + \dots$ [√]

$\geq 1 + nk/100$ (since all terms are positive, as $k > 0$ and $n > 0$). [H]

(c) (i)

$$\left(1 + \frac{k}{100}\right)^n = 1 + \frac{nk}{100} + \binom{n}{2}\frac{k^2}{100^2} + \binom{n}{3}\frac{k^3}{100^3} + \dots$$ [E]

$$= 1 + \frac{nk}{100} + \frac{n(n-1)k^2}{20000} + \frac{n(n-1)(n-2)k^3}{6000000} + \ldots$$ [√]

(c) (ii) Given $k = 3$, required to prove :

$$\frac{nk}{100} < \frac{n(n-1)k^2}{20000}$$

$$\frac{3n}{100} < \frac{9n(n-1)}{20000}$$ [E]

$$200 < 3(n-1) \qquad \text{(as } n > 0\text{)}$$ [H]

$$200 < 3n - 3$$

$$203 < 3n$$

$67.6 < n$, so $68 \le n$, as n must be an integer. [H]

(d) Consider the account consisting of n parts, each being an investment account as in (a).
The first part is invested for n years,
the second part is invested for (n-1) years,
... and the last part is invested for 1 year.
So, from (a), we get a total payout of :
$A(1+ k/100) + A(1+ k/100)^2 + A(1+ k/100)^3 + \ldots + A(1+ k/100)^n$. [√]
This is a geometric *series*, with a first term of $a = A(1+ k/100)$,
and a common ratio of $r = (1+ k/100)$. [H]
We can find its sum to n terms, s_n, from the formula :

$$s_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{A\left(1+\frac{k}{100}\right)\left(1 - \left(1+\frac{k}{100}\right)^n\right)}{1-\left(1+\frac{k}{100}\right)}$$ [H]

$$= \frac{A\left(1+\frac{k}{100}\right)\left(\left(1+\frac{k}{100}\right)^n - 1\right)}{\frac{k}{100}}$$

$$= A\frac{100}{k}\left(1+\frac{k}{100}\right)\left(\left(1+\frac{k}{100}\right)^n - 1\right)$$

$$= A\frac{100+k}{k}\left(\left(1+\frac{k}{100}\right)^n - 1\right)$$ [H]

(e) <u>First consider investment (a)</u>.

Real value after n years $= A\left(1+\frac{k}{100}\right)^n\left(1 - \frac{d}{100}\right)^n$

$$= A\left(1+\frac{k}{100}\right)^n\left(1 - \frac{k}{100}\right)^n, \quad \text{as } k = d.$$

$$= A\left[\left(1+\frac{k}{100}\right)\left(1 - \frac{k}{100}\right)\right]^n.$$

$$= A\left[1 - \frac{k^2}{10000}\right]^n$$ [H]

But $1 - \frac{k^2}{10000} < 1$, as $\frac{k^2}{10000} > 0$. [H] and $1 - \frac{k^2}{10000} > 0$, as $0 < d < 100$, (given).

$$\left[1 - \frac{k^2}{10000}\right]^n < 1$$ [H]

$$\Rightarrow \quad A\left[1 - \frac{k^2}{10000}\right]^n < A$$ Thus the customer has lost money on this investment.

Now consider the planned savings account (d).
This consists of n investment accounts of type (a), as noted above in part (d). Each of these will lose money by the proof just given. Hence the entire account must lose money. [H]

Question 6

My instructions to the markers :
This is a problem-solving question. The main goal is to work out the direction required in order to win the competition. Some students, unfamiliar with Kiwi water sports, may have had difficul understanding the story. However, they could still obtain a good mark by following the mathematical tasks stated. Be understanding, even if some of the interpretations are rather exotic. Because the emphasis in problem-solving is on reaching the right answer, a certain amount of informal reasoning, and leaps which miss out what are to us important steps, should be allowed. This is the hardest question in the exam. Those who work at it should be able to earn at least 10 marks, but I am only expecting a median of about 6.

How the students got on :

Median :	7
Upper quartile :	11
Lower quartile :	4
Number scoring 20 :	2

What the markers thought :
All candidates seemed to understand the general idea of this question, but found the technical level of difficulty formidable. (a) was started well, but many omitted the mandatory derivative test. Most could form the correct equation for (b), but only a minority realised this needed to be manipulated into a quadratic. Techniques for solving trig equations could well be taught more vigorously. (c) was usually answered well. Some vague answers were given to (d), candidate not realising the need to relate this to their earlier answer to (a). Only the very best candidates came to grips with (e), and some of these even messed up their units, or shied at the last differentiation.

The marking scheme :

(a) To find the maximum speed, let $s'(ø) = 0$.

$$s'(\phi) = 5(\,2\sin(2\phi) + \sin(\phi)) = 0$$ [E]

$$4\sin(ø)\cos(ø) + \sin(ø) = 0$$

$$\sin(ø)\,[4\cos(ø) + 1] = 0$$ [H]

So, <u>either :</u> $\sin(ø) = 0$

and hence $ø = 0$ or 180^o [0 or π (or 2π ...)]

<u>or :</u> $\cos(ø) = -0.25$

and hence $ø = 104.48^o$ [1.8235 radians etc]. [√]

<u>Second derivative test</u> :

$$s''(\phi) = 5(\,4\cos(2\phi) + \cos(\phi))$$

$s''(0) = 25 > 0$, hence $ø = 0$ gives a minimum speed

$s''(104.48^o) = -18.75 < 0$, hence $ø = 104.48^o$ gives a maximum speed. [H]

$s(104.48^o) = 10.625$ kph This is the maximum speed. [√]

It is attained when $ø = 104.48^o$. [√]

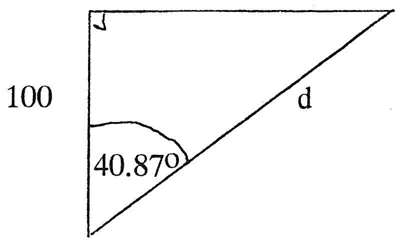
(b) Let $5(1 - \cos(2ø) - \cos(ø)\,) = 0.5$ [E]

$$\begin{aligned} 1 - \cos(2\phi) - \cos(\phi) &= 0.1 \\ \cos(2\phi) + \cos(\phi) - 0.9 &= 0 \qquad [H] \\ 2\cos^2(\phi) - 1 + \cos(\phi) - 0.9 &= 0 \\ 2\cos^2(\phi) + \cos(\phi) - 1.9 &= 0 \end{aligned}$$

Applying the quadratic formula : $\cos(\phi) = 0.75623$ (disregarding $\cos(\phi) = -1.256$) [√]

Hence $\phi = 40.87^o$ [0.71326 radians] [√]

(c)



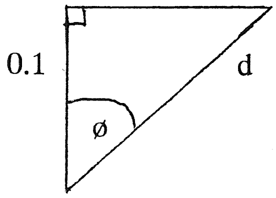
The shortest distance, d, will occur at the smallest possible value for ø, which we have just found to be 40.87°. [E]

Hence $\cos(40.87^o) = 100/d$ [E]

$d = 132$ metres. [√]

(d) At the fastest speed, the competitor would be sailing away from the finishing line. [E]

(e) In general,



$d = 0.1 / \cos(\phi)$ [d is distance in km]

$t = d / s$ [t is time in hours]

$t = 0.1/[\cos(\phi).5(1 - \cos(2\phi) - \cos(\phi)]$ [H]

$$\frac{dt}{d\phi} = \frac{-0.1[-5\sin(\phi)(1 - \cos(2\phi) - \cos(\phi)) + 5\cos(\phi)(2\sin(2\phi) + \sin(\phi))]}{[\cos(\phi).5.(1 - \cos(2\phi) - \cos(\phi))]^2} \qquad [H]$$

At the shortest time, we require $dt/d\phi = 0$.

Thus :

$$\begin{aligned} -5\sin(\phi)(1 - \cos(2\phi) - \cos(\phi)) + 5\cos(\phi)(2\sin(2\phi) + \sin(\phi)) &= 0 \\ -5\sin(\phi)[1 - \cos(2\phi) - \cos(\phi)] + 5\cos(\phi)[4\sin(\phi)\cos(\phi) + \sin(\phi)] &= 0 \\ 5\sin(\phi)[-1 + \cos(2\phi) + \cos(\phi)] + 5\cos(\phi)\sin(\phi)[4\cos(\phi) + 1] &= 0 \\ 5\sin(\phi)[-1 + \cos(2\phi) + \cos(\phi) + \cos(\phi)(4\cos(\phi) + 1)] &= 0 \qquad [H] \end{aligned}$$

So either : $5\sin(\phi) = 0$, giving $\phi = 0$ or 180^o , both of which are infeasible,

or :

$$\begin{aligned} -1 + \cos(2\phi) + \cos(\phi) + \cos(\phi)(4\cos(\phi) + 1) &= 0 \\ -1 + 2\cos^2(\phi) - 1 + \cos(\phi) + 4\cos^2(\phi) + \cos(\phi) &= 0 \\ 6\cos^2(\phi) + 2\cos(\phi) - 2 &= 0 \end{aligned}$$

Applying the quadratic formula : $\cos(\phi) = 0.43426$ (-0.76759 is infeasible)

$\phi = 64.26^o$ (1.12158 radians) [H]

And when $\phi = 64.26^o$, $t = 0.038745$ hours (2.3 minutes) [√]

<u>First derivative test :</u>

When $\phi = 64^o$, $dt/d\phi = -0.1[-4.49397(1.17729) + 2.19(2.4748)] / [\quad]^2 < 0$.

When $\phi = 64.26^o$, $dt/d\phi = 0$

When $\phi = 65^o$, $dt/d\phi = -0.1[-4.53154(1.22017) + 2.113(2.4384)] / [\quad]^2 > 0$.

Hence 2.3 minutes is the shortest time possible to complete the competition, and this is achieved when $\phi = 64.26^o$. [H]

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