NZEST

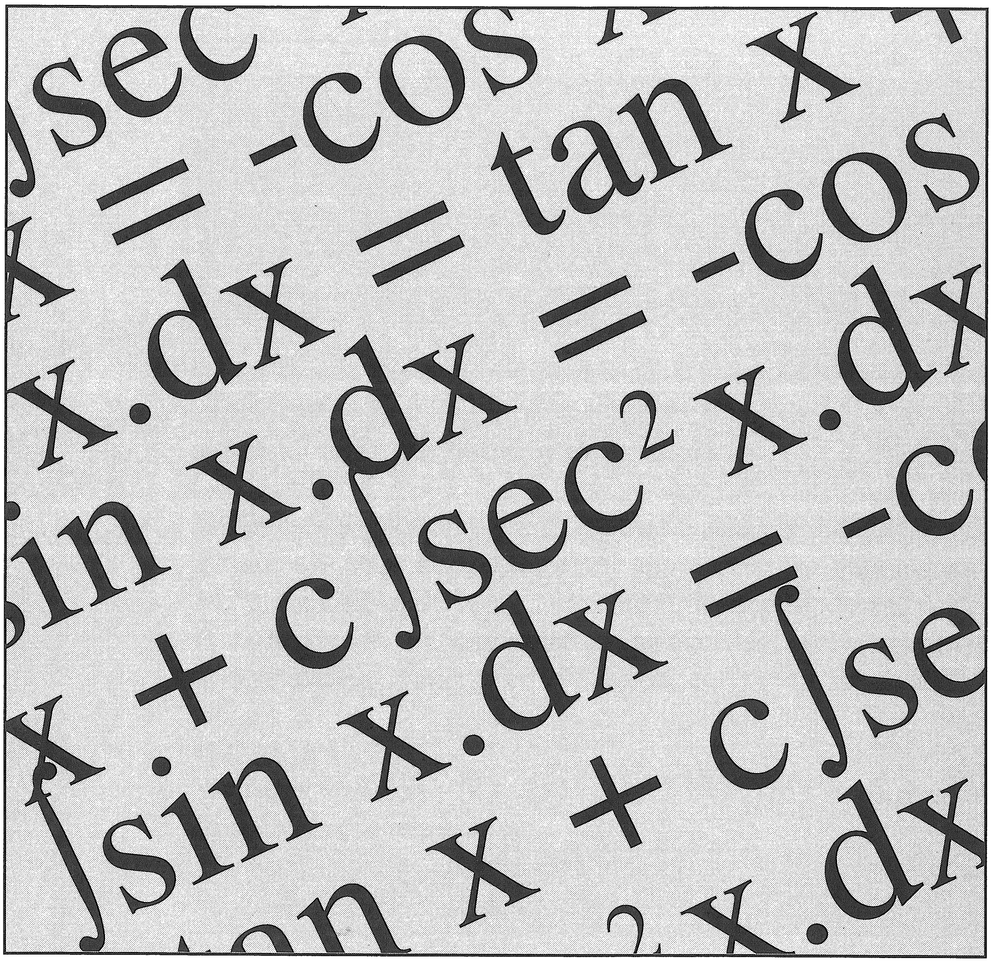
NEW ZEALAND EDUCATION
& SCHOLARSHIP TRUST

ACADEMIC ACHIEVEMENT BY CHOICE

NZEST SCHOLARSHIP EXAMINATION

1997 EXAMINER'S REPORT AND SOLUTIONS

MATHEMATICS WITH CALCULUS



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NZEST Mathematics with Calculus Examiners' Report 1997

This year a slight change has occurred with fewer problem-solving questions (2) and more mathematical extension questions (3). This did not seem to bother any candidates or teachers. Otherwise the format and intention of the paper remains the same. The emphasis is still to give each student the opportunity to demonstrate their ability and creativity in a variety of innovative and challenging situations, rather than to merely systematically assess competence across a syllabus. Most students appeared to enjoy their work. All those scoring over 40 are to be congratulated as we know this was a searching exam.

Some teachers are still getting unnecessarily frustrated with the length of the paper as they imagine we expect all students to write full answers to all 6 questions. In fact we expect a good student to answer the equivalent of 5 questions (which could well amount to full answers to 4 and part answers to the remaining 2) as indicated by the marking scheme which awards 20 marks per question. Only about 10 students with raw totals over 94 had to be adjusted slightly to give a top mark bounded by 100%. For every other student their actual mark in this paper is exactly the same as their raw total. It is not easy to compose creative questions for this type of paper which are completely fair to all students. We believe that this paper structure provides valuable protection as it allows a student to omit several sections of questions without undue penalty if these are inadvertently inappropriate to that particular student. The option of altering the instructions to "attempt any 5 questions" would restrict the students freedom of choice and require the examiners to make all questions of equal difficulty.

Exam Statistics

Number sat :	776									
Median :	56									
Upper quartile :	68									
Lower quartile :	43									
Percentage :	0-	10-	20-	30-	40-	50-	60-	70-	80-	90-
Frequency :	0.5	1.6	4.4	10.5	14.8	26.9	18.6	13.4	5.7	3.5

These results are very commendable. It is especially pleasing to note very few candidates ending up in the "tail" of the distribution. A generally high standard of algebraic skills was noted by the markers.

In the marking scheme which follows marks are classified in our customary categories :
E : an easy mark awarded for any approximate or vaguely equivalent working,
M : a medium mark requiring an accurate and moderately difficult response, and
H : a hard mark only awarded for an excellent answer showing high skill or deep understanding. Refer the 1996 examiners' report for a fuller discussion.

Question 1
Statistics : median : 14, LQ : 11, UQ : 16, number scoring 20 : 19
Instructions to markers :

This is intended to test basic technical skills. The expected median mark is 14. Be generous assessing alternative methods.

(a) f(4) = 9 [M] [1]

(b) $f'(4) = \lim_{h \to 0}\left[\frac{\left((4+h)^2 - 4(4+h)+3\right)^2 - 9}{h}\right]$ [M]

$= \lim_{h \to 0}\left[\frac{\left(16+8h+h^2-16-4h+3\right)^2 - 9}{h}\right]$ [E]

$= \lim_{h \to 0}\left[\frac{\left(h^2+4h+3\right)^2 - 9}{h}\right]$ [H]

$= \lim_{h \to 0}\left[\frac{h^4+8h^3+22h^2+24h+9-9}{h}\right]$

$= \lim_{h \to 0}\left[h^3+8h^2+22h^2+24\right]$

= 24 [M] [4]

(c) $f'(x) = 2(x^2-4x+3)(2x-4)$ [E] [M]

$f'(4) = 2(16-16+3)(8-4) = 24$ [M] [3]

(d) y - 9 = 24(x - 4) [E]
y = 24x - 87 or 24x -y - 87 = 0 [M] [2]

(e) Slope of normal = -1/24 [E]
Equation 24(y-9) = - (x-4) [E]
x + 24y = 220 [M] [3]

(f) $f'(1) = 2(1-4+3)(2-4) = 0$ [M]

$f'(3) = 2(9-12+3)(6-4) = 0$ [M]

As $(x^2-4x+3) = (x-1)(x-3)$ it follows that $f(x) = (x-1)^2 (x-3)^2$.
So $f'(x)$ must have both (x-1) and (x-3) as factors. [H]
Hence $f'(1) = f'(3) = 0$. [3]

(g) The centre of the circle must lie on the normal line in (e). [E]
Letting y be any integer other than 9 in that equation

and then solving for x yields a centre. For example, if y = 10 then x = -20.
In this case the centre is (-20, 10). [M]
The radius will be the distance from (-20, 10) to (4, 9) which is $\sqrt{577}$. [H]
Thus the circle has equation $(x+20)^2 + (y-10)^2 = 577$. [M] [4]

Markers' comments :
Part (a) was very well done. Part (b) caused problems, especially to students who tried to find f′(x) instead of f′(4). Parts (c-e) were well done. Many failed to give a satisfactory explanation in part (f).

Question 2
Statistics : median : 10, LQ : 6, UQ : 13, top mark : 20
Instructions to markers :

This is a harder technical question leading into matters of extension and interpretation that require some original thinking. Expected median is about 10 marks.

(a) Let $u = x^n$. Thus $\dfrac{du}{dx} = nx^{n-1}$ [E]

Thus $\int x^{n-1}e^{-x^n}\,dx = \int x^{n-1}e^{-u}\dfrac{dx}{du}\,du$ [M]

$= \int x^{n-1}e^{-u}\left(\dfrac{1}{nx^{n-1}}\right)du$

$= \int e^{-u}\left(\dfrac{1}{n}\right)du$

$= -\dfrac{1}{n}e^{-u} + C$ [M]

$= -\dfrac{1}{n}e^{-x^n} + C$ [M] [4]

Do not deduct a mark if "+ C" missing.

(b) (i) $\displaystyle\int_0^2 xe^{-x^2}\,dx = \left[-2e^{-x^2}\right]_0^2$ [M]

$= 0.5 - 0.5e^{-4} = 0.491$ [M] [2]

(ii) $\displaystyle\int_0^2 xe^{-x^2}\,dx \approx \frac{1}{4}\left[f(0)+2(f(0.5)+f(1)+f(1.5))+f(2)\right]$ [E]

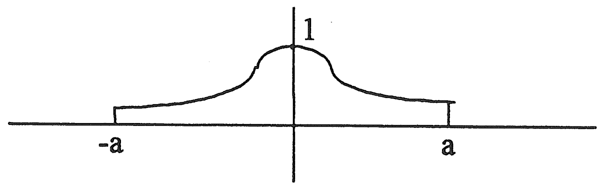
$= 0.25[0 + 2(0.39 + 0.37 + 0.16) + 0.04]$ [M]
$= 0.47$ [M] [3]

(iii) $\int_0^2 xe^{-x^2}\,dx \approx \frac{1}{6}[f(0)+4f(0.5)+2f(1)+4f(1.5)+f(2)]$ [E]

$= \frac{1}{6}[0 + 1.56 + 0.74 + 0.64 + 0.04]$ [M]

$= 0.50$ [M] [3]

(c)



[E] shape
[M] coords [2]

(d) method 1 : using a hollow cylinder as the integrating device

$\int_0^a 2\pi.xy.dx = \int_0^a 2\pi.xe^{-x^2}.dx$ [E]

$= 2\pi\int_0^a xe^{-x^2}.dx$

$= 2\pi\left[-\frac{1}{2}e^{-x^2}\right]_0^a$ [H]

$= \pi\left[1-e^{-a^2}\right]$ [M] [3]

(d) method 2 : using a horizontal circular cross-section as the integrating device
This requires the addition of the volume of the base cylinder.

volume = $\int_{e^{-a^2}}^{1} \pi(-\ln(y))dy + \pi.a^2e^{-a^2}$ [E]

$= -\pi[y\ln y - y]\ -\pi[y\ln y - y]_{e^{-a^2}}^{1} \quad + \quad \pi.a^2e^{-a^2}$ [H]

$= \pi\ \pi.\left[-a^2e^{-a^2} - e^{-a^2}\right] \quad + \quad \pi.a^2e^{-a^2}$

$= \pi\left[1-e^{-a^2}\right]$ [M] [3]

(e) The region is symmetric about the y-axis [E]
So the part to the left of the y-axis traces out exactly the same solid as the part to the right. [H] [2]

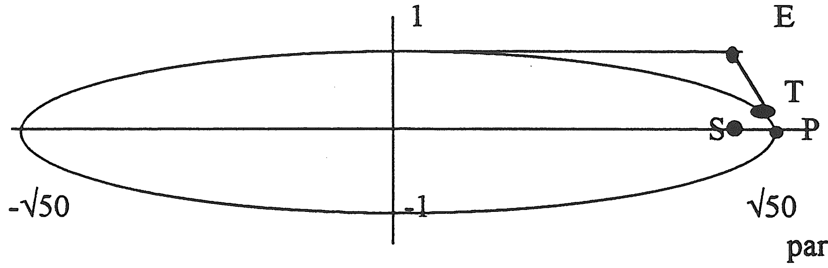
(f) $\lim_{a \to \infty} \pi\left[1 - e^{-a^2}\right] = \pi$ [M] [1]

Markers' comments :
In (a) the better students integrated by inspection and scored 4. The substitution was not well done by many. In (b) many students could not manage the formal method of integration by substitution, but probably would have been able to determine the answer by inspection which raises the question as to the sense of requesting a formal method. In (d) many wrote down an expression but did not evaluate it. Most did not account for the volume of the base cylinder. Many did not seem to understand what they were doing with the volume of revolution method.

Question 3
Statistics : median : 12, LQ : 10, UQ : 15, number scoring 20 : 9
Instructions to markers :

This is an application question. Students who make a genuine attempt to apply what they know about the ellipse should be rewarded. Do not penalise algebraic deviations unduly. Allow plenty of follow-on marks. No rigour needed. Expected median 10.

(a) & (b) and part (e)



(a) any ellipse [E]
intercepts [M]
(b) E [E]
S [M]
P [E]
part (e) tangent TE [E]
[6]

(c) Substitute (7,1) into the equation of the ellipse [E]

$$7^2 + 50 \times 1^2 = 99$$ [E] [E]

(7,1) is outside the ellipse since $99 > 50$ [H] [4]

(d) The line EP is $\dfrac{y-1}{x-7} = \dfrac{-1}{\sqrt{50}-7}$ [E]

$\Leftrightarrow (\sqrt{50}-7)y - (\sqrt{50}-7) = -x + 7$

$$\Leftrightarrow x = \sqrt{50} - (\sqrt{50}-7)y$$ [M]

Substituting into ellipse equation :

$$\left(\sqrt{50}-(\sqrt{50}-7)y\right)^2+50y^2=50$$ [E]

$50 - 2\sqrt{50}\,(\sqrt{50} - 7)y + (\sqrt{50} - 7)^2\, y^2 + 50y^2 = 50$

So $y = 0$ or $y = 0.0201$ [M]

Thus the intersection points are $(\sqrt{50}, 0)$ and $(7.0696, 0.0201)$ [M] [5]

(e) One mark is awarded for drawing the tangent TE in the diagram [E] (see (a))
(i) Take the equation of the ellipse and differentiate implicitly w.r.t. x :

$$2x+100y\frac{dy}{dx}=0\ ,$$

$$\text{so}\quad \frac{dy}{dx}=\frac{-2x}{100y}$$

Substituting (u,v) : $\dfrac{dy}{dx}=\dfrac{-u}{50v}$ [M] [1]

(ii) Slope of ET is $\dfrac{1-v}{7-u}$ [M] [1]

(iii) These slopes must be the same, so $\dfrac{1-v}{7-u}=\dfrac{-u}{50v}$ [M]

$$u^2 - 7u = 50v - 50v^2$$

$$u^2 + 50v^2 = 50v + 7u \quad (*)$$

(u,v) lies on the ellipse, so $u^2 + 50v^2 = 50$ (**) [H]

subtract (**) from (*) : $50v + 7u - 50 = 0$

$$v=\frac{50-7u}{50}$$

substitute back into the equation of the ellipse :

$$u2+\ 50\left(\frac{50-7u}{50}\right)^2=50$$

$$50u2 + 2500 - 700u + 49u2 = 2500$$

$$u = 7.0707$$

and so $v = 0.0101$ and the point is $(7.0707, 0.0101)$ [H] [3]

Markers' comments :

Successful use of surds such as $\sqrt{50}$ was rare. In (b) generous allowance was made for placing S and P due to difficulties with the elongated ellipse. In (c) a brief but correct explanation was accepted in lieu of a strict proof. Some attempted very difficult alternative methods here. In (d) some made the problem meaningless by excessive (sometimes incorrect) rounding of decimals. Most ignored the hint. In (e) the implicit differentiation was well done.

Question 4

Statistics : median : 10, LQ : 8, UQ : 14, number scoring 20 : 8

Instructions to markers :

This is a test of the student's ability to reason and demonstrate logical powers. It should be marked with attention to detail. The median mark is expected to be 8.

(a) If $x = 1/3$, then $\dfrac{\sin(\pi.x)}{x(1-x)} = 3.897$ [E] [E]

and $\pi < 3.897 < 4$

If $x = 3/4$, then $\dfrac{\sin(\pi.x)}{x(1-x)} = 3.771$ [E] [E]

and $\pi < 3.771 < 4$ [4]

(b) (i) $f'(x) = \dfrac{x(1-x)\pi\cos(\pi.x) - (1-2x)\sin(\pi.x)}{x^2(1-x)^2}$ [E] [M] [2]

(ii) Let $x = 1/2$ [E]

$f'(0.5) = \dfrac{0.5(1-0.5)\pi\cos(0.5\pi) - (1-1)\sin(0.5\pi)}{0.5^2(1-0.5)^2} = 0$ [E] [M]

So $x = 1/2$ is a turning point , [E]

and thus (using given information) it is the only turning point on (0, 1). [4]

(iii) Apply the 1st derivative test.

x	0.4	0.5	0.6
$f'(x)$	0.743	0	-0.743

[M]

This implies that the turning point when $x = 0.5$ is a maximum. [H]

And when $x = 0.5$, $f(x) = 4$

Thus the unique turning point is a maximum at (0.5, 4)

Thus $\dfrac{\sin(\pi.x)}{x(1-x)} \le 4$ on the interval $(0,1)$ [H] [3]

(c) (i) $\lim_{x\to 0}\left(\frac{\sin(\pi.x)}{x(1-x)}\right) = \pi \lim_{x\to 0}\left(\frac{\sin(\pi.x)}{\pi.x}\right) \lim_{x\to 0}\left(\frac{1}{1-x}\right)$ [E]

$= \pi.1.1 = \pi$ [H] [2]

(ii) To find $\lim_{x\to 1}\left(\frac{\sin(\pi.x)}{x(1-x)}\right)$ let u = 1-x

$$\lim_{x\to 1}\left(\frac{\sin(\pi.x)}{x(1-x)}\right) = \lim_{u\to 0}\left(\frac{\sin(\pi.(1-u))}{u(1-u)}\right)$$ [H]

$$=\pi \lim_{u\to 0}\left(\frac{\sin(\pi.(1-u))}{\pi.u}\right) \lim_{u\to 0}\left(\frac{1}{1-u}\right)$$

$$=\pi \lim_{u\to 0}\left(\frac{\sin(\pi.(1-u))}{\pi.(1-u)}\right) \lim_{u\to 0}\left(\frac{1}{1-u}\right)$$ (using the limit given) [H] [2]

$= \pi.1.1 = \pi$

(iii) Using (b) (ii) and the hint, the maximum must be attained at x = 0 or x = 1 [H]

$f(0) = f(1) = \pi$

Thus $f(x) \geq \pi$ [H] [2]

(iv) Since for the extended function $f(x)$ we have $\pi \leq f(x) \leq 4$ for $0 \leq x \leq 1$,

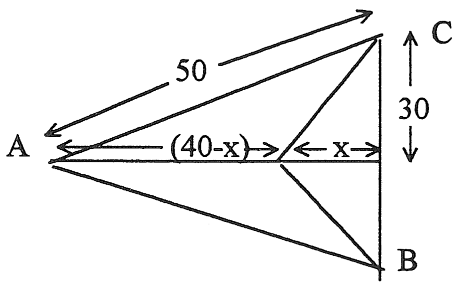
it follows that for the original function $f(x)$ we have
$\pi \leq f(x) \leq 4$ for $0 < x < 1$. [H] [1]

Markers' comments :
Parts (a), (bi) and (bii) combined to give 10 marks provided students could calculate, use the quotient rule, and recognise that (bii) could be done by substitution. In (biii) too many attempted the very difficult 2nd derivative. In (ci) and (cii) attention to detail and logical presentation were needed. Some students made good use of L'Hospital's rule. Parts (ciii) and (civ) were marked hard and required clear reasoning. Candidates should be encouraged to provide clear explanations for steps taken and to indicate implications accurately when asked to prove something.

Question 5
Statistics : median : 6, LQ : 4, UQ : 10, number scoring 20 : 2
Instructions to markers :

This is a problem solving question. Some generous marks will be given for brave students who make an attempt. The H marks may only be earned by a genuine argument, though the question does NOT ask for a completely rigorous proof. Expected median is about 7.

(a)



[E][E]

length of motorway = f(x) = 40-x + 2 $\sqrt{x^2 + 900}$ [M]

$f'(x) = -1 + (x^2 + 900)^{-0.5}.2x$ [M]

let $f'(x) = 0$ [E]

$\Rightarrow 2x = \sqrt{x^2 + 900}$

$\Rightarrow 4x^2 = x^2 + 900$

$\Rightarrow 3x^2 = 900 \quad \Rightarrow x = 10\sqrt{3}$ [M]

$f''(x) = 2.(x^2 + 900)^{-0.5} - x(x^2 + 900)^{-1.5}.2x$

$f''(10\sqrt{3}) = 2.(300 + 900)^{-0.5} - (300 + 900)^{-1.5}.600$

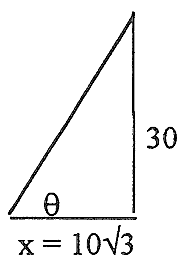
$= 0.04334$

which is positive. So this is a local minimum. [H]

f(10√3) = 40 + 30√3 [M]

Thus the minimum length of motorway is 40 + 30√3 km. [8]

(b)



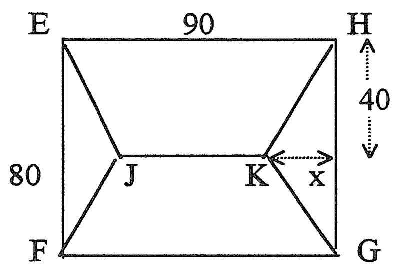
$\tan\theta = \frac{30}{10\sqrt{3}} = \sqrt{3}$ [E]

so θ = π/3

Thus ∠ADC = ∠ADB = ∠BDC = 2π/3 [E] [2]

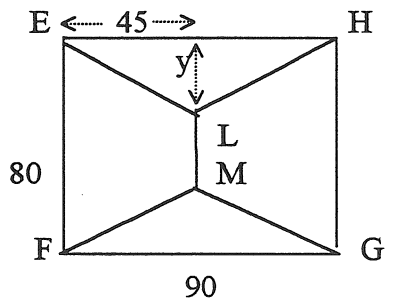
(c) Any 2 cases are acceptable, but the following is recommended :

case 1



[E]

case 2



[E]

Let J, K, L, M be Steiner points.

Thus in case (1) $\tan(\pi/3) = 40 / x$

Hence $x = 40/\sqrt{3}$

Thus $HK = 80/\sqrt{3}$

and total length = $320/\sqrt{3} + (90 - 80/\sqrt{3})$

$= 90 + 240/\sqrt{3} = 90 + 80\sqrt{3}$ km [M]

Similarly, in case (2) $\tan(\pi/3) = 45 / y$

Hence $y = 45/\sqrt{3}$

Thus $HK = 90/\sqrt{3}$

and total length = $360/\sqrt{3} + (80 - 90/\sqrt{3})$

$= 80 + 270/\sqrt{3} = 80 + 90\sqrt{3}$ km [M]

NB up to this point candidate scores full marks in part (c) for any feasible pair of road plans.

The shortest possible length is thus $90 + 80\sqrt{3}$ km [M] and case 1 above is the best possible plan [M]

because :

1. we have assumed the optimal solution will be symmetric [H]

2. we have used Steiner points as in part (a) [H]

3. case 1 and 2 above are the only possible ways to satisfy 1 & 2 [H]

4. $90 + 80\sqrt{3} < 80 + 90\sqrt{3}$ [H]

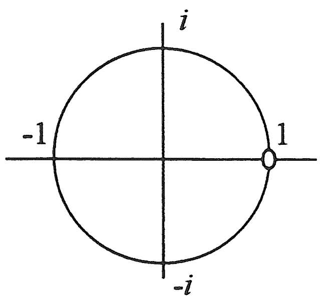
Alternatively, the student could repeat the calculus routines as in (a) [10]

Markers' comments :
Part (a) was either well done or poorly done. Very few tested to see if the turning point was a maximum or minimum. The algebra looked difficult, but those who got into it handled it well. Students should be reminded that if a question mentions a maximum or a minimum then calculus is likely to be involved. There were a number of innovative methods that scored well here. Part (b) was often ignored even though it was straightforward and the marks quite generous. Part (c) was challenging, but those who had a go gained the first 4 marks easily. The few students completing the question produced a variety of highly original methods, some stunning in their clarity of thought. Two noteworthy quotes both explaining why their answers did not match that given in the question:
"this is different from that given, which shows that your answer is wrong," and
"close enough for government work".

Question 6
Statistics : median : 5, LQ : 1, UQ : 7, number scoring 20 : 1
Instructions to markers :

This is a challenging question testing a high level of technical skill and also the student's ability to follow an investigative lead, linking ideas from various parts of the question to develop new concepts. Expected median is about 6.

(a) (i)



or in words "all points on the unit circle excluding (1,0)" [E] [M] [2]

(ii) $f(z) = f(\cos\theta + i\sin\theta) = 1 + \dfrac{4}{(\cos\theta - i\sin\theta) - 1}$ [E]

$$= 1+4\left(\frac{-1}{2}\left(1-i\cot\frac{\theta}{2}\right)\right)$$ (using given formula) [M]

$$= -1 + 2\,i\cot\frac{\theta}{2}$$ [M] [3]

(iii) Required to show A(1, 0), B(cos θ, sin θ) and C(-1, 2 $\cot\frac{\theta}{2}$) are collinear.

Slope of AB $= \dfrac{\sin\theta}{\cos\theta - 1}$ [E]

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\left(1-2\sin^2\frac{\theta}{2}\right)-1}$$

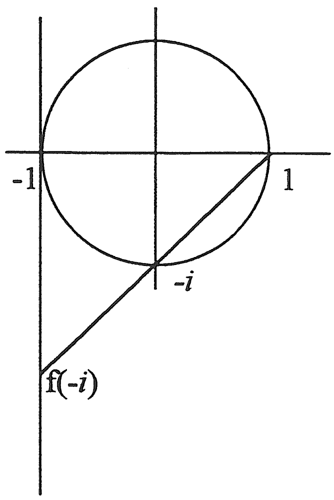
$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2}}$$ [H]

$$= -\cot\frac{\theta}{2}$$

Slope of AC $= \dfrac{2\cot\frac{\theta}{2}}{-1-1} = -\cot\dfrac{\theta}{2}$ [M] [3]

Thus ABC are collinear.

(iv)



[E]

[M] [2]

(b) $\omega^4 = 1$

Let $\omega = r \text{ cis } \theta$

Thus $(r \text{ cis } \theta)^4 = 1$

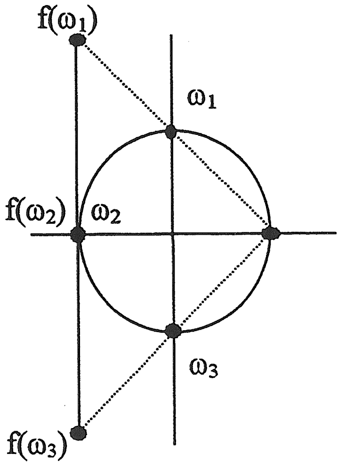
so $r^4 \text{ cis } (4\theta) = 1$ [E]

hence $r = 1$

and $\theta = 0, \pi/2, \pi, -\pi/2.$ [M]

Hence solutions are $\{ \text{cis } 0, \text{ cis } \pi/2, \text{ cis } \pi, \text{ cis}(-\pi/2) \}$ ie $\{1, i, -1, -i.\}$ [M] [3]

(c)



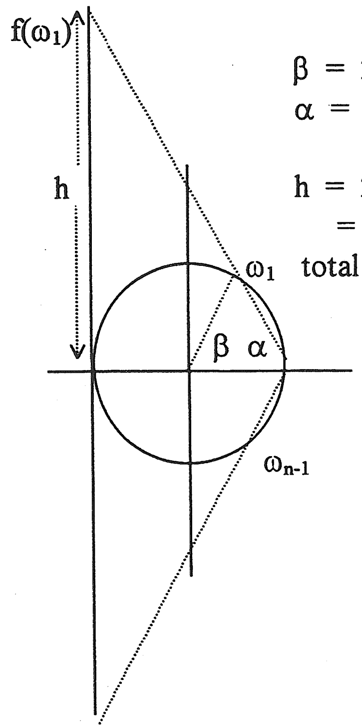
the 4-th roots [E]

$f(\omega_1)$ [M]
$f(\omega_2)$ [M]
$f(\omega_3)$ [M] [4]

(d) All $f(\omega_j)$ are on the line $x = -1$ from (a) (ii). [H]

The extreme points will be $(-1, f(\omega_1))$ and $(-1, f(\omega_{n-1}))$

that is $(-1, f(\text{cis}(2\pi/n))$ and $(-1, f(\text{cis}(-2\pi/n))$



$\beta = 2\pi/n$
$\alpha = \pi/2 - \pi/n$ [H]

$h = 2 \tan(\pi/2 - \pi/n)$
$= 2 \cot(\pi/n)$

total length of line segment $= 2h$
$= 4 \cot(\pi/n)$

[H] [3]

Markers' comments :

For most students this was their last question and answers reflected the consequent fatigue and time pressure. In part (a)(i) a reasonable number realised that there is a problem when $z = 1$, but many thought this meant an asymptote was involved. Very few realised the domain of this function involves a circle. The better students completed part (a)(ii) well. Only about 20 good answers were obtained for part (a)(iii). Many candidates seemed not to understand the word "collinear". Part (b) was the most productive part the result being familiar to many. In part (c) confused labelling of points caused trouble.

Feedback from school evaluations :

Most felt that the degree of difficulty was appropriate and were generally supportive, some saying it was our "best yet". Specific comments included :
a good mix of problem solving and extension questions,
some parts required excessive time, such as expanding $(x^2-4x+3)^2$ in question 1,
objections to $\int \ln y \, dy$ some thought necessary in question 2(d),
not enough trigonometry,
question 6 should have been placed earlier in the paper,
question 5 should have had more hints,
not enough differential equations,
could we have less words, and an easier introduction to each question, and
too many questions were sequential, with second half depending on earlier work.

Thanks for this helpful advice which will be noted with regard to next year's paper.

Rory Barrett,
Professor David Gauld,
Eddie Mak,
Alastair McNaughton