NZEST MATHEMATICS WITH CALCULUS

1998 SOLUTIONS

Question 1.

(a) Suppose that $y = x(x+1)^2$.

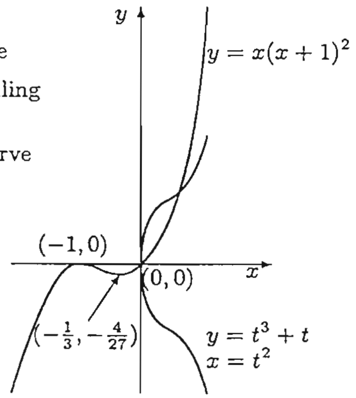
(i) Find where $\dfrac{dy}{dx}$ is positive and where $\dfrac{dy}{dx}$ is negative. [3]
$\frac{dy}{dx} = (x+1)^2 + 2x(x+1) = (x+1)(3x+1) = 3(x+1)(x+\frac{1}{3})$ $\surd$.
Thus $\frac{dy}{dx}$ is positive when either $x < -1$ or $x > -\frac{1}{3}$ $\surd$ and negative when $-1 < x < -\frac{1}{3}$ $\surd$.

(ii) Find where the curve meets the x-axis. [2]
When $x(x+1)^2 = 0$ $\surd$. Thus when $x = 0$ or -1 $\surd$.

(iii) Draw x and y axes on a sheet of graph paper with $-2 \le x \le 1$ and $-2 \le y \le 4$. Then, using the information from (i) and (ii), sketch (do NOT plot) the graph of $y = x(x+1)^2$. Label the important points you found in (i) and (ii). [2]

(a)(iii) $\surd$ for curve
$\surd$ for labelling

(c)(ii) $\surd\surd$ for curve



(b) Consider the parametric equations $x = t^2$ and $y = t^3 + t$ as t ranges through the real numbers.

(i) Determine $\dfrac{dy}{dx}$ in terms of t. [2]
$\frac{dy}{dt} = 3t^2 + 1$ and $\frac{dx}{dt} = 2t$ $\surd$ so $\frac{dy}{dx} = \frac{3t^2+1}{2t}$ $\surd$.

(ii) Hence show that the slope of the curve defined by these equations tends to $\pm\infty$ as t tends either to 0 or to $\pm\infty$. [2]
As t tends to 0 the denominator tends to 0 while the numerator tends to 1 so $\frac{dy}{dx}$ tends to $\pm\infty$ $\surd$.
For $t \ne 0$ we have $\frac{dy}{dx} = \frac{3t+1/t}{2}$, which tends to $\pm\infty$ as t tends to $\pm\infty$ $\surd$.

(c) (i) Show that elimination of the parameter t from the equations $x = t^2$ and $y = t^3 + t$ leads to the cartesian equation $y^2 = x(x+1)^2$. [2]
$y^2 = t^2(t^2+1)^2 (\surd) = x(x+1)^2$ $\surd$.

(ii) Using (a), (b) and (c)(i), or otherwise, and on the same sheet of graph paper as in (a)(iii), sketch the graph of the curve defined by $x = t^2$ and $y = t^3 + t$ for $-1 \le t \le 1$. Be sure to label the two graphs so that it is clear which answers (a)(iii) and which answers (c)(ii). [2]
Answer with (a)(iii) above.

(d) (i) Write down a formula which gives the distance from the point $(a, 0)$ $(a > 0)$ to the point $(t^2, t^3 + t)$. [2]

Let the distance be Δ. Then $\Delta^2 = (t^2 - a)^2 + (t^3 + t)^2$ $\surd\surd$.

(ii) Show that if $a \leq \frac{1}{2}$ then there is exactly one point on the curve defined by $x = t^2$ and $y = t^3 + t$ which is nearest to $(a, 0)$ but if $a > \frac{1}{2}$ then there are two such points. Your answer should include an explanation as to why these points are nearest to $(a, 0)$. [3]

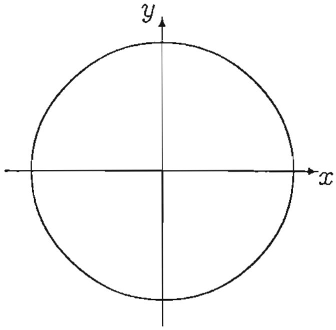
To find the nearest point we need $\frac{d\Delta}{dt} = 0$, equivalently, $\frac{d\Delta^2}{dt} = 0$. Thus we need $2(t^2 - a)2t + 2(t^3 + t)(3t^2 + 1) = 0$, ie $2t(3t^4 + 6t^2 + 1 - 2a) = 0$ $\surd$. This means that either $t = 0$ or $t^2 = \frac{-3 \pm \sqrt{9 - 3(1 - 2a)}}{3}$. The only way that the last can have a real root is for the term within the square root to be at least 9, ie for $1 - 2a$ to be non-positive, ie $a \geq 1/2$. When $a = 1/2$ the only real root (a triple root) is $t = 0$, and so for $a \leq 1/2$ there is a single nearest point. When $a > 1/2$ the equation $t^2 = \frac{-3 \pm \sqrt{9 - 3(1 - 2a)}}{3}$ has two real solutions, one positive and the other negative; by symmetry the corresponding points on the curve will be equidistant from the point $(0, a)$, ie there are two nearest points $\surd$. That the origin is nearest when $a \leq 1/2$ and the other two points are nearest when $a > 1/2$ follows from the fact that for large $|t|$, Δ is large and decreasing as $|t|$ decreases and as there is a single critical value when $a \leq 1/2$ this must be a minimum. If $a > 1/2$, Δ must decrease as t decreases from a large value, going to a minimum at the positive root of the equation $\frac{d\Delta^2}{dt} = 0$ then it must increase again as t continues on to 0; thus in this case there are two nearest points $\surd$.

Question 2.

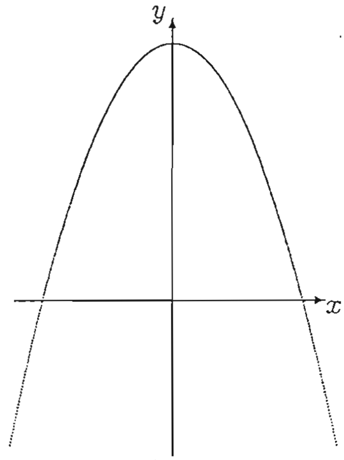
John and Marsha are old friends. John is now postmaster on a space station orbiting a planet in a distant galaxy. Marsha is now the pilot of an IGNZP (Inter Galactic New Zealand Post) mail delivery starship.

The space station orbits the planet in a circle and the path of the starship lies in the same plane as this circle. In this question we are interested only in where the orbit of the space station and the path of the starship meet or cross. Taking a set of axes with origin the centre of the planet, the orbit of the space station can be represented by $x^2 + y^2 = R^2$ and the path of the starship by $y = k - x^2$ where $k > 0$.

(a) Sketch the graph of $x^2 + y^2 = 4$. [1]



(a) √



(b) √

(b) Sketch the graph of $y = 4 - x^2$. [1]

(c) (i) John and Marsha can fail to meet or they can meet at a number of points. What is the maximum number of points at which John and Marsha can meet? [1]
The parabola can cut the circle in at most 4 points √.

(ii) For each possibility from 0 up to the maximum possible, sketch a separate diagram showing how the orbit and path meet or cross. [3]

0 1 2 2 3 4

√ for one correct diagram, √√ for three correct diagrams, √√√ for all six correct diagrams.

(d) In each case where there are exactly **three** points of contact

(i) Find the coordinates where John and Marsha can make contact in terms of R. [5]
In this case the curves touch at $(0, R)$, so $y = k - x^2$ gives $k = R$ √. Then

$x^2 = R - y$ so $x^2 + y^2 = R^2$ becomes $R - y + y^2 = R^2$ $\surd$, ie $y^2 - y + R - R^2 = 0$. We already know that $y = R$ is a solution, so $y - R$ is a factor: by inspection we have $y^2 - y + R - R^2 = (y - R)(y - 1 + R)$ $\surd$. Thus $y = 1 - R$ is the other solution for y, so that $x = \pm\sqrt{2R - 1}$ $\surd$. Thus the three points of contact are $(-\sqrt{2R-1}, 1 - R)$, $(0, R)$ and $(\sqrt{2R-1}, 1 - R)$ $\surd$.

(ii) Show that in this case $R > \frac{1}{2}$. [2]

For the solutions to be real we must have $2R - 1 \geq 0$ so $R \geq 1/2$ $\surd$. However if $R = 1/2$ then all three roots coincide, so we must have $R > 1/2$ $\surd$.

(e) In the case where there are exactly **two** points of contact and the circle is entirely within the parabola let a point of contact be (x_1, y_1).

(i) Show that $y_1 = \frac{1}{2}$ no matter what the value of R is. [4]

Substituting $x^2 = k - y$ into $x^2 + y^2 = R^2$ yields $y^2 - y + (k^2 - R^2) = 0$ $\surd$. By symmetry, for there to be two points of contact we must have just a single solution of this equation $\surd$, so that it must be a perfect square and in that case the solution y_1 must be negative half of the coefficient of y $\surd$; hence $y_1 = 1/2$ $\surd$.

(ii) Show that Marsha must have set her path so that $k = R^2 + \frac{1}{4}$. [3]

From the reasoning above we must have $k - R^2 = (1/2)^2$ $\surd$ $\surd$, so we have $k = R^2 + \frac{1}{4}$ $\surd$.

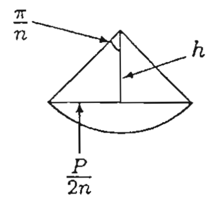
Question 3.

(a) Two students, John and Hemi, argue about the areas formed by bending the same piece of wire into various regular polygons. Hemi says that an equilateral triangle has a smaller area than a square which in turn has a smaller area than a regular pentagon. Determine if Hemi is correct. [12]
Suppose that the wire has length l √.

- If the wire is shaped into an equilateral triangle then each side has length $\frac{l}{3}$ √. The triangle has base $\frac{l}{3}$ and height $\frac{1}{2}\frac{l}{3}\tan\frac{\pi}{3} = \frac{l\sqrt{3}}{6}$ √. Thus the area of the triangle is $\frac{1}{2}\frac{l}{3}\frac{l\sqrt{3}}{6} = \frac{l^2\sqrt{3}}{36}$ √.
- If the wire is shaped into a square then each side has length $\frac{l}{4}$ √ and the area is $\frac{l^2}{16}$ √.
- If the wire is shaped into a regular pentagon then each side has length $\frac{l}{5}$ √. One can think of the pentagon as being made up of 5 isosceles triangles having base of length $\frac{l}{5}$ and angle $\frac{2\pi}{5}$ √; then the height of the triangle is $\frac{l}{10}\cot\frac{\pi}{5}$ √. Thus each triangle has area $\frac{1}{2}\frac{l}{5}\frac{l}{10}\cot\frac{\pi}{5} = \frac{l^2}{100}\cot\frac{\pi}{5}$ and hence the pentagon has area $\frac{l^2}{20}\cot\frac{\pi}{5}$ √.

Now we need to compare $\frac{l^2\sqrt{3}}{36}$, $\frac{l^2}{16}$ and $\frac{l^2}{20}\cot\frac{\pi}{5}$ √. From a calculator we have $\frac{\sqrt{3}}{36} = 0.0481$, $\frac{1}{16} = 0.0625$ and $\frac{1}{20}\cot\frac{\pi}{5} = 0.0688$ √. Thus the pentagon has the greatest area and the triangle the least.

(b) Maria claims that any regular polygon has an area less than the area of a circle of the same perimeter. She bases her argument on the fact that $x < \tan x$ for $0 < x < \frac{\pi}{2}$.
By using this fact or any other method prove her hypothesis. [8]
Suppose that the perimeter is P √. For the circle, this means that the radius is $\frac{P}{2\pi}$, so the area of the circle is $\pi(\frac{P}{2\pi})^2 = \frac{P^2}{4\pi}$ √.
Suppose that the polygon has n sides. Then it is made up of n isosceles triangles each having base of length $\frac{P}{n}$ and opposite angle $\frac{2\pi}{n}$ √.



From the diagram we have $h = \frac{P}{2n}\cot\frac{\pi}{n}$ √, so the triangle has area $\frac{P^2}{4n^2}\cot\frac{\pi}{n}$ √ and hence the polygon has area $\frac{P^2}{4n}\cot\frac{\pi}{n}$ √.

We need to show that $\frac{P^2}{4n}\cot\frac{\pi}{n} < \frac{P^2}{4\pi}$ √. On rearrangement and cancellation this is equivalent to $\frac{\pi}{n} < \tan\frac{\pi}{n}$ √, which is the inequality given to us with $x = \frac{\pi}{n}$ as the latter lies strictly between 0 and $\frac{\pi}{2}$.

Question 4.

(a) (i) Solve the equation $x + \sqrt{1-x^2} = \sqrt{2}$. [3]
Squaring each side of $\sqrt{1-x^2} = \sqrt{2} - x$ $\surd$ gives $1 - x^2 = 2 + x^2 - 2\sqrt{2}x$ $\surd$ so that $2x^2 - 2\sqrt{2}x + 1 = 0$, and hence $x = \frac{1}{\sqrt{2}}$ $\surd$.

(ii) Hence or otherwise solve the equation $\sin x + \cos x = \sqrt{2}$ for $-\pi \leq x \leq \pi$. [2]
Setting $y = \sin x$ the equation becomes $y + \sqrt{1-y^2} = \sqrt{2}$ $\surd$ so from (i) we get $y = \frac{1}{\sqrt{2}}$, ie $\sin x = \frac{1}{\sqrt{2}}$, so $x = \frac{\pi}{4}$ $\surd$. Note that even though $\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$, this is not a solution because $\cos\frac{3\pi}{4}$ is negative!

(b) Define the function $\sec x$ in terms of $\cos x$ being careful to state any values of x for which your definition is not valid. [1]
$\sec x = \frac{1}{\cos x}$, this being valid as long as $\cos x$ is non-zero, ie x is not of the form $(n + \frac{1}{2})\pi$ $\surd$.

(c) Using the definition given in (b) prove that $\sec^2 x = 1 + \tan^2 x$. [2]
We have $\sin^2 x + \cos^2 x = 1$ $\surd$ so, on dividing through by $\cos^2 x$ we get $\tan^2 x + 1 = \sec^2 x$ $\surd$.

(d) Using (c) or otherwise prove that $\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}$. [4]
Set $y = \cos^{-1} x$. Then from (c) we obtain
$\tan^2 y = \sec^2 y - 1 = (\dfrac{1}{\cos y})^2 - 1$ $\surd$ $= \frac{1}{x^2} - 1$ $\surd$ $= \frac{1-x^2}{x^2}$ $\surd$. Now taking the square root and noting that when x is negative so is $\tan(\cos^{-1} x)$, we obtain $\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}$ $\surd$.

(e) Find an expression for z in terms of x which does not have any trigonometric function in it if $\sin^{-1} x = \tan^{-1} z$. [Hint: one approach is to use $\tan(\sin^{-1} x)$.] [3]
One method: As $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ $\surd$ it follows that $\tan(\sin^{-1} x) = \cot(\cos^{-1} x)$ $\surd$ $= \frac{1}{\tan(\cos^{-1} x)} = \frac{x}{\sqrt{1-x^2}}$ $\surd$.

Another method: Let $\theta = \sin^{-1} x = \tan^{-1} z$. Then $\sin\theta = x$ and $\tan\theta = z$ $\surd$. Taking similar triangles in which one angle is a rightangle and another is θ, where in one triangle the side opposite the angle θ has length x so the hypotenuse has length 1 and in the other triangle the side opposite the angle θ has length z so the side next to the angle has length 1 $\surd$ we deduce that in the first triangle the side next to the angle θ has length $\sqrt{1-x^2}$, so that $z = \frac{x}{\sqrt{1-x^2}}$ $\surd$.

(f) Note that it is unlikely that a solution to the following makes use of parts (a) to (e). A hillside is in the shape of a plane sloping down from west to east and making an angle of 10° with the horizontal. A straight road goes across and up the hillside making an angle of 6° with the horizontal on the hillside. Figure (i) shows a vertical west-east slice through the hillside. Figure (ii) shows the hillside from above and slightly to the east. Neither figure is drawn to scale.

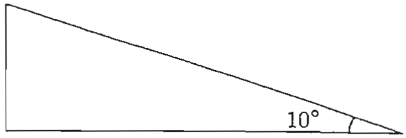


Figure (i)

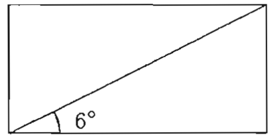


Figure (ii)

Copy Figure (iii) to whatever size is convenient. The plane shown is a horizontal plane. On it sketch how the line of the road lies on the hillside in 3 dimensions. Start by drawing the hillside using this plane as a base.

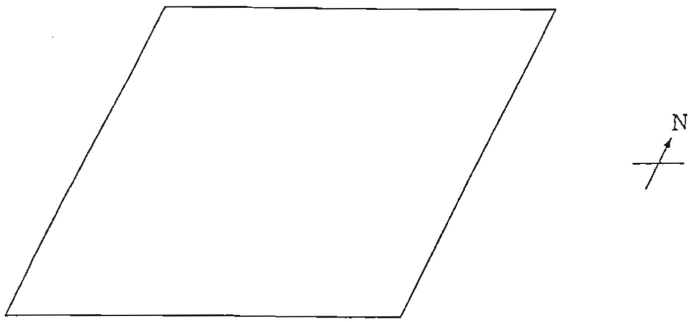
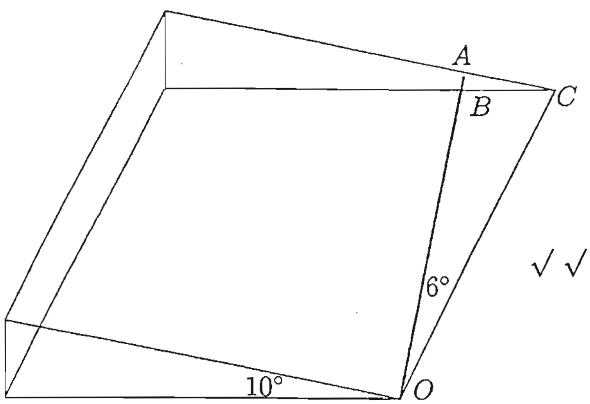


Figure (iii)

Find the angle of elevation of the road. [5]

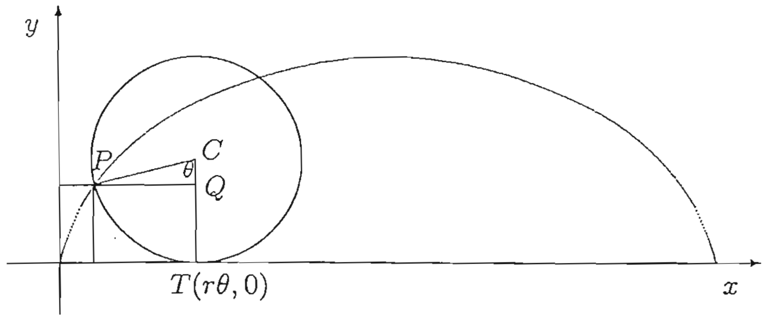


In the diagram, the road is labelled OA. OC is the line of intersection of the horizontal plane and the hillside and B is the point on the horizontal plane vertically below A. The angles ABO, ACO and BCO are right-angles.

To find the angle of elevation of the road we need to find the angle AOB, call it $\alpha°$. Suppose that $|OC| = l$. From triangle ACO we have $|OA| = l \sec 6°$ and $|AC| = l \tan 6°$ $\surd$. From triangle ABC we have $|AB| = |AC| \sin 10° = l \tan 6° \sin 10°$ $\surd$. Thus $\sin \alpha° = \frac{|AB|}{|OA|} = \frac{l \tan 6° \sin 10°}{l \sec 6°} = \sin 6° \sin 10°$, so $\alpha = 1.04$ $\surd$.

Question 5.

(a) A wheel of radius r rolls along the x-axis beginning so that it is touching the x-axis at the origin. A point P is fixed on the circumference of the wheel and was at the origin. The curve traced out by P is called a cycloid. The diagram below shows the situation when the wheel has rotated through θ radians, and includes the first arch of the cycloid itself.



We confine our attention to the case where $0 \leq \theta \leq 2\pi$, in which case P traces out one arch of the cycloid as shown.

(i) Show that the coordinates of the point $P(x, y)$ are given by

$$x = r\theta - r\sin\theta, \qquad y = r - r\cos\theta. \qquad [3]$$

$x = |OT| - |PQ| = r\theta - r\sin\theta \checkmark \checkmark$ and $y = |CT| - |CQ| = r - r\cos\theta \checkmark$.

(ii) Find $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ and hence show that $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}$. [3]

$\frac{dx}{d\theta} = r - r\cos\theta \checkmark$ and $\frac{dy}{d\theta} = r\sin\theta \checkmark$ so $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2} \checkmark$.

(iii) The area under the arch is $\displaystyle\int_0^{2\pi r} y\,dx$, which transforms to $\displaystyle\int_0^{2\pi} y\frac{dx}{d\theta}d\theta$ for the parameter θ. Calculate this area giving your answer in terms of π. [3]

Area $= \int_0^{2\pi} r^2(1-\cos\theta)^2 d\theta \checkmark = r^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta$

$= r^2\int_0^{2\pi}(1 - 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2})d\theta \checkmark = 3\pi r^2 \checkmark$.

(iv) Deduce from (iii) that the circle centred at $(\pi r, r)$ of radius r, which you may assume fits within the arch, divides the arch into 3 regions of equal area. [2]

By symmetry the region to the left of the circle has the same area as the region to the right of the circle $\checkmark$. Since the circle has area πr^2, it follows that the other two regions also have area $\pi r^2 \checkmark$.

(v) The length of the arch is $\displaystyle\int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$. Calculate this length showing all working. [3]

Length $= \int_0^{2\pi}\sqrt{(r - r\cos\theta)^2 + (r\sin\theta)^2}d\theta \checkmark = r\int_0^{2\pi}\sqrt{(2 - 2\cos\theta)}$

$= r\int_0^{2\pi} 2\sin\frac{\theta}{2}d\theta \checkmark = 2r[-2\cos\frac{\theta}{2}]_0^{2\pi} = 8r \checkmark$.

(b) By choosing suitable expressions for x and n in the expansion of $(1+x)^n$ prove the following results. It might help you to begin by writing out the expressions in full without the Σ sign.

(i) $\sum_{k=0}^{n}\binom{n}{k}=2^n$. [1]
Setting $x=1$ in the expansion $(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$ we get $\sum_{k=0}^{n}\binom{n}{k}=2^n$ immediately $\surd$.

(ii) $\sum_{k=1}^{p}\binom{2p}{2k-1}=1+\sum_{k=1}^{p}\binom{2p}{2k}$. [2]
Setting $x=-1$ and $n=2p$ we get $0=\sum_{k=0}^{2p}(-1)^k\binom{2p}{k}\surd$
$=\sum_{k=0}^{p}\binom{2p}{2k}-\sum_{k=1}^{p}\binom{2p}{2k-1}=1+\sum_{k=1}^{p}\binom{2p}{2k}-\sum_{k=1}^{p}\binom{2p}{2k-1}$
so $\sum_{k=1}^{p}\binom{2p}{2k-1}=1+\sum_{k=1}^{p}\binom{2p}{2k}$ $\surd$.

(iii) $\sum_{k=1}^{2m}(-1)^{k-1}\binom{4m}{2k-1}=0$. [Hint: You may wish to use a complex number for x in this part.] [3]
Setting $x=i$ and $n=4m$ we get $(1+i)^{4m}=\sum_{k=0}^{4m}i^k\binom{4m}{k}$ $\surd$. Now $(1+i)^4$ is real, so equating imaginary parts of the expansion we have $0=\sum_{k=1}^{2m}i^{2k-1}\binom{4m}{2k-1}$ $\surd$
so $0=\sum_{k=1}^{2m}i^{2(k-1)}\binom{4m}{2k-1}$ and hence $\sum_{k=1}^{2m}(-1)^{k-1}\binom{4m}{2k-1}=0$ as $i^2=-1$ $\surd$.

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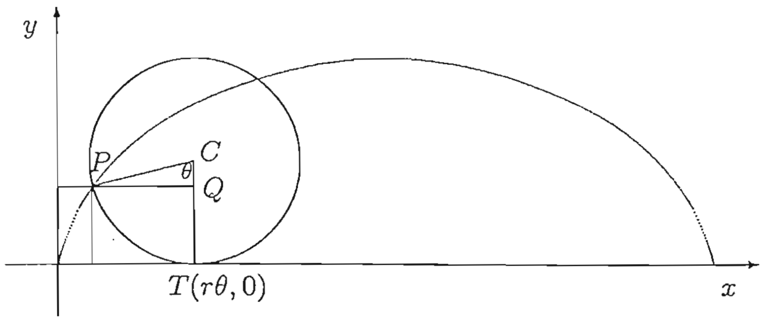

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Question 6.

The number π, the ratio of the circumference of a circle to its diameter, is of fundamental importance in the modern world. Over the past few millenia attempts have been made to express it as a rational number such as 3 or 22/7 and, more recently, to find rational numbers which are arbitrarily close to π. This question investigates the approximation of π by 22/7 and then goes on to look at one way of finding closer rational approximations.

Consider the rational function $\dfrac{x^4(1-x)^4}{x^2+1}$.

On expanding the numerator and denominator and then dividing we obtain:

$$\frac{x^4(1-x)^4}{x^2+1} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}.$$

(a) Evaluate $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx$. You may assume that $\displaystyle\int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4}$, which can be verified easily using the substitution $x = \tan\theta$ but you are not required to do this here. [3]

$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx = \int_0^1 (x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1})dx \checkmark$
$= [\frac{x^7}{7} - \frac{4x^6}{6} + x^5 - \frac{4x^3}{3} + 4x]_0^1 - 4\int_0^1 \frac{1}{x^2+1}dx \checkmark = \frac{22}{7} - \pi \checkmark$.

(b) Show that $0 < \displaystyle\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx < \frac{1}{2^8}$ as follows:

(i) Explain why $\dfrac{x^4(1-x)^4}{x^2+1} \geq 0$ for all x. [1]

For all x we have $x^4(1-x)^4 \geq 0$ and $x^2+1 > 0$, so that $\frac{x^4(1-x)^4}{x^2+1} \geq 0 \checkmark$.

(ii) Show that for all x strictly between 0 and 1, $\dfrac{x^4(1-x)^4}{x^2+1} > 0$. From this it follows that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx > 0$ but you are not asked to prove this. [1]

When x is strictly between 0 and 1 we have $x^4(1-x)^4 > 0$, so $\frac{x^4(1-x)^4}{x^2+1} > 0$ for such $x \checkmark$.

(iii) Explain why $\dfrac{x^4(1-x)^4}{x^2+1} < x^4(1-x)^4$ for all x strictly between 0 and 1. [2]

For x strictly between 0 and 1 we have $x^2+1 > 1 \checkmark$, so
$\frac{x^4(1-x)^4}{x^2+1} < \frac{x^4(1-x)^4}{1} = x^4(1-x)^4 \checkmark$.

(iv) By finding the maximum of $x^4(1-x)^4$ on the interval from 0 to 1, show that

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx < \frac{1}{2^8}.$$

[3]

The maximum of $x^4(1-x)^4$ on the interval is reached when $x = \frac{1}{2} \checkmark$ EITHER by symmetry OR by differentiation we get $\frac{d(x^4(1-x)^4)}{dx} = 4x^3(1-x)^3(1-2x)$ which is 0 when $x = \frac{1}{2}$. Thus the maximum value is $\frac{1}{2^8}$ $\surd$, so that $\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx < \int_0^1 \frac{1}{2^8}dx = \frac{1}{2^8}$ $\surd$.

(c) (i) What does the result in (b) tell us about approximating π by 22/7? [2]

While π is not equal to $\frac{22}{7}$ $\surd$, it is within $\frac{1}{2^8}$ $\surd$.

(ii) Explain the significance of the number 0 in its first appearance in $0 <$

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx < \frac{1}{2^8}.$$ [1]

This means that $\frac{22}{7}$ is strictly greater than π $\surd$.

(d) Using the result (considered in question 5(b)(iii)) that $\sum_{i=1}^{2m}(-1)^{i-1}\binom{4m}{2i-1} = 0$ or otherwise prove that the polynomial $P(x) = \sum_{i=1}^{2m}\binom{4m}{2i-1}x^{2m+i-1}$ is divisible by $x+1$. [2]

The polynomial $P(x)$ is divisible by $x+1$ if and only if $P(-1) = 0$ $\surd$.

Now $P(-1) = \sum_{i=1}^{2m}(-1)^{2m+i-1}\binom{4m}{2i-1} = \sum_{i=1}^{2m}(-1)^{i-1}\binom{4m}{2i-1} = 0$ by 5(b)(iii) $\surd$.

(e) Deduce from (d) that when we divide the polynomial $x^{4m}(1-x)^{4m}$ by x^2+1 the remainder does not contain a term in x. [5]

[Hint: You may write $x^{4m}(1-x)^{4m} = Q(x) + R(x)$, where $Q(x)$ and $R(x)$ are polynomials in x and $Q(x)$ contains only terms of even degree while $R(x)$ contains only terms of odd degree. Show that division of $Q(x)$ by x^2+1 yields only a constant remainder and that $R(x) = -xP(x^2)$, where $P(x)$ is the polynomial in part (d).]

With $Q(x)$ and $R(x)$ as described in the hint we see that division of $Q(x)$ by x^2+1 involves only even powers of x, so all terms must involve only even powers of x and in particular the remainder must be constant $\surd$. As $(1-x)^{4m} = \sum_{i=0}^{4m}\binom{4m}{i}(-x)^i = \sum_{i=0}^{2m}\binom{4m}{2i}x^{2i} - \sum_{i=1}^{2m}\binom{4m}{2i-1}x^{2i-1}$ $\surd$, it follows that $R(x) = -\sum_{i=1}^{2m}\binom{4m}{2i-1}x^{4m+2i-1} = -\sum_{i=1}^{2m}\binom{4m}{2i-1}(x^2)^{2m+i-1}x = -xP(x^2)$ $\surd$. In (d) we saw that $P(x)$ is divisible by $x+1$, so that $P(x^2)$ is divisible by x^2+1, and hence $R(x)$ is also divisible by x^2+1, ie there is no remainder term when $R(x)$ is divided by x^2+1 $\surd$. Thus when we divide the polynomial $x^{4m}(1-x)^{4m}$ by x^2+1 the remainder does not contain a term in x $\surd$.

Comment: One can use the result in (e) to obtain an inequality like that of (b) which, in turn, leads to a more accurate rational approximation to π than 22/7.