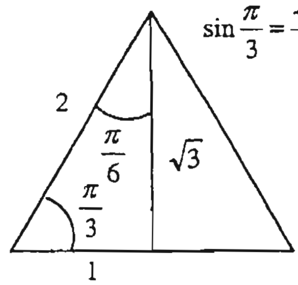
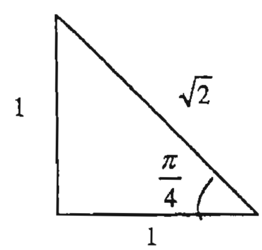
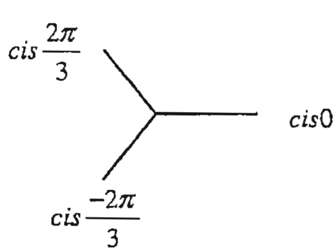
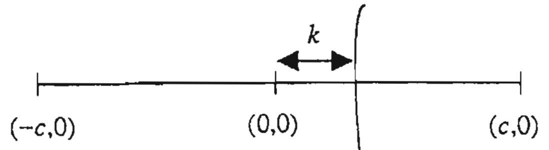
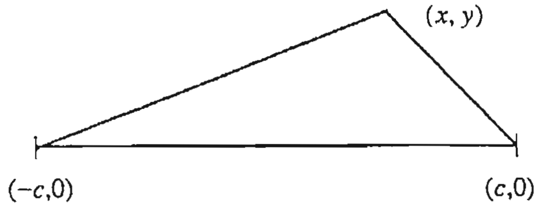
NZEST Mathematics with Calculus

1999 Solutions

1(a)(i)	 $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ $\quad\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ $\quad\tan\frac{\pi}{3}=\sqrt{3}$	The marks are for showing an understanding of the process by drawing the triangle 1 for triangle divided with lengths shown 1 for angles shown SUBTOTAL 2
1(a)(ii)		1 mark for the correct triangle with sides & angles shown SUBTOTAL 1
1(b)(i)	$z^3=1=cis0$ $z^3=cis2k\pi$ $z=1^{\frac{1}{3}}cis\frac{0}{3},\ 1^{\frac{1}{3}}cis\frac{2\pi}{3},\ 1^{\frac{1}{3}}cis\frac{-2\pi}{3}$ $z=cis\frac{0}{3},\ cis\frac{2\pi}{3},\ cis\frac{-2\pi}{3}$	1 mark each for first and last line. (Accept $\frac{4\pi}{3}$) 1 mark for a middle line demonstrating understanding in some way. Two examples given SUBTOTAL 3
1(b)(ii)		SUBTOTAL 1

1(b)(iii)	$\left(z^2+1\right)^3=1$ $z^2+1=cis0,\ or\ z^2+1=cis\frac{2\pi}{3},\ or\ z^2+1=cis\frac{-2\pi}{3}$ if $z^2+1=cis0$, $z^2+1=1$ so $z=0$ if $z^2+1=cis\frac{2\pi}{3}$, $z^2+1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ $z^2=-\frac{3}{2}+\frac{\sqrt{3}}{2}i$ $=\sqrt{3}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ $=\sqrt{3}cis\left(\frac{5\pi}{6}\right)$ $z=3^{\frac{1}{4}}cis\left(\frac{5\pi}{12}\right),\ 3^{\frac{1}{4}}cis\left(\frac{-7\pi}{12}\right)$ Since complex roots of a polynomial over the real numbers come in conjugate pairs, the roots corresponding to the case $z^2+1=cis\frac{-2\pi}{3}$ must be $z=3^{\frac{1}{4}}cis\left(\frac{-5\pi}{12}\right),\ 3^{\frac{1}{4}}cis\left(\frac{7\pi}{12}\right)$	1 mark 1 mark 1 mark 1 mark 1 mark 1 mark SUBTOTAL 6
1(c)(i)	$1+i=\sqrt{2}cis\frac{\pi}{4}$	1 mark SUBTOTAL 1
1(c)(ii)	$(1+i)^{4n}=\left[\sqrt{2}^4 cis\frac{4\pi}{4}\right]^n$ $=(-1)^n 4^n$ which is real	1 mark 1 mark SUBTOTAL 2
1(d)(i)	$(1+i)^{4n}=1+\binom{4n}{1}i+\binom{4n}{2}i^2+\binom{4n}{3}i^3+........+i^{4n}$ $=1+\binom{4n}{1}i-\binom{4n}{2}-\binom{4n}{3}i+.......+i^{4n}$	1 mark NB. Don't worry about general term 1 mark SUBTOTAL 2
1(d)(ii)	As $(1+i)^{4n}$ is real $\binom{4n}{1}-\binom{4n}{3}+\binom{4n}{5}......=0$ So $\sum_{j=1}^{2n}\binom{4n}{2j-1}(-1)^j=0$	1 mark 1 mark SUBTOTAL 2

2(a)(i)	 When the comet meets the line connecting the two planets, the distance from the furthest point is $c+k$ and the distance from the closest is $c+k$. So $(c+k)+(c-k)=d$ $d=2k$ Since $x<c \quad d<2c$	1 mark 1mark SUBTOTAL 2
2(a)(ii)	 $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=d$	1mark for a diagram 1mark SUBTOTAL 2
2(a)(iii)	$\sqrt{(x+c)^2+y^2}=d+\sqrt{(x-c)^2+y^2}$ $(x+c)^2+y^2=d^2+(x-c)^2+y^2+2d\sqrt{(x-c)^2+y^2}$ $4xc-d^2=2d\sqrt{(x-c)^2+y^2}$ $16x^2c^2-8d^2xc+d^4=4d^2[x^2-2xc+c^2+y^2]$ $16x^2c^2-8d^2xc+d^4=4d^2[x^2-2xc+c^2+y^2]$ $(16c^2-4d^2)x^2-4d^2y^2=4d^2c^2-d^4$ $(16c^2-4d^2)x^2-4d^2y^2=d^2(4c^2-d^2)$ $\dfrac{4x^2}{d^2}-\dfrac{4y^2}{4c^2-d^2}=1$	1 mark 1 mark 1 mark 1 mark 1 mark 1 mark SUBTOTAL 6
2(a)(iv)	One branch of a hyperbola	1 mark SUBTOTAL 1
2(b)(i)	$\dfrac{4}{d^2}=\dfrac{1}{25}$ So $d=10$	1 mark SUBTOTAL 1
2(b)(ii)	$\dfrac{4}{4c^2-100}=\dfrac{1}{200}$ $c^2=225$ $c=\pm 15$ Coords of planets are $(-15,0)$, $(15,0)$	 1 mark SUBTOTAL 1

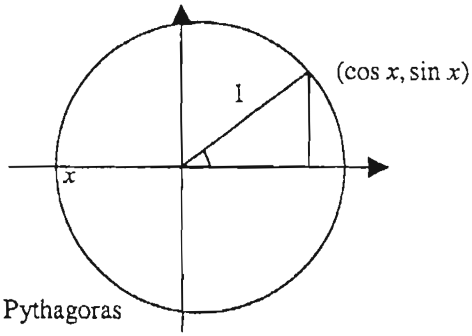
2(c)(i)	$\cos\theta = \dfrac{x-3}{2} \qquad \sin\theta = \dfrac{y-1}{3}$ $\left(\dfrac{x-3}{2}\right)^2 + \left(\dfrac{y-1}{3}\right)^2 = 1$	1 mark 1 mark SUBTOTAL 2
2(c)(ii)	$\dfrac{dy}{d\theta} = 3\cos\theta$ $\dfrac{dx}{d\theta} = -2\sin\theta$ So gradient of tangent: $\dfrac{dy}{dx} = \dfrac{3\cos\theta}{-2\sin\theta} = -\dfrac{3}{2}$	 1 mark 1 mark SUBTOTAL 2
2(c)(iii)	The tangent touches at $\left(3+\dfrac{2}{\sqrt{2}}, 1+\dfrac{3}{\sqrt{2}}\right)$ or (4.41,3.12) The equation of the normal is $y-1-\dfrac{3}{\sqrt{2}} = \dfrac{2}{3}\left(x-3-\dfrac{2}{\sqrt{2}}\right)$ When the normal cuts the x-axis $y=0$ $-1-\dfrac{3}{\sqrt{2}} = \dfrac{2}{3}\left(x-3-\dfrac{2}{\sqrt{2}}\right)$ $-1-\dfrac{3}{\sqrt{2}} = \dfrac{2}{3}x-2-\dfrac{4}{3\sqrt{2}}$ $\dfrac{2x}{3} = 1-\dfrac{5}{3\sqrt{2}}$ $x = \dfrac{3}{2}-\dfrac{5}{2\sqrt{2}}$ Required coords are $\left(\dfrac{3}{2}-\dfrac{5}{2\sqrt{2}}, 0\right)$ or (-0.27,0)	1 mark 1 mark 1 mark SUBTOTAL = 3

3(a)(i)	$f'(x)=\dfrac{(e^x-1)e^x-(e^x+1)e^x}{(e^x-1)^2}$ $=\dfrac{-2e^x}{(e^x-1)^2}$ As $e^x \neq 0$ $f(x)$ has no turning points	1 mark 1 mark SUBTOTAL 2
3(a)(ii)	Vertical asymptote at $e^x-1=0$ This occurs when $x=0$	1 mark SUBTOTAL 1
3(a)(iii)	As $x\to\infty$, $f(x)\to 1$ and as $x\to-\infty$, $f(x)\to -1$	1 mark SUBTOTAL 1
3(a)(iv)	$f'(x)=\dfrac{-2e^x}{(e^x-1)^2}$ As $x\to-\infty$, $e^x\to 0$, so $f'(x)\to 0$ As $x\to\infty$, $f'(x)\to\dfrac{-2e^x}{(e^x)^2}=\dfrac{-2}{e^x}\to 0$	1 mark 1 mark SUBTOTAL 2
	So Hemi is correct	1 mark SUBTOTAL 1
3(b)	$g(x)=\dfrac{2}{x}\sin x^2$ $g'(x)=-\dfrac{2}{x^2}\sin x^2+4\cos x^2$ As $x\to\pm\infty$ $g'(x)$ will oscillate between -4 and 4 .	2 marks 1 mark for "oscillate" or equivalent 1 mark for bounds SUBTOTAL 4
	Marsha is correct	1 mark SUBTOTAL 1
3(c)(i)	$h'(x)=\dfrac{(2x^2+3)2x-(x^2+1)4x}{(2x^2+3)^2}$ $=\dfrac{2x}{(2x^2+2)^2}$ For $x>0$, $2x>0$ Also $2(x^2+3)^2>0$ Thus $h'(x)>0$ and hence h is increasing.	2 marks 1 mark 1 mark SUBTOTAL 4
3(c)(ii)	$h(x)=\dfrac{x^2+1}{2x^2+3}=\dfrac{1+\dfrac{1}{x^2}}{2+\dfrac{3}{x^2}}\to\dfrac{1}{2}$ as $x\to\infty$	1 mark SUBTOTAL 1

3(c)(iii)	If $x > k \Rightarrow h(x) > c - \frac{1}{100}$ Then if we choose k such that $h(k) > c - \frac{1}{100}$ then this will automatically follow as h is increasing for $x > 0$ $\frac{k^2+1}{2k^2+3} > \frac{1}{2} - \frac{1}{100}$ $100k^2 + 100 > 98k^2 + 147$ $2k^2 > 47$ $k > \sqrt{23.5}$	1 mark 1 mark 1 mark SUBTOTAL 3

4(a)	$\frac{\Delta V}{V}\times 100\% = 1\%$ $\Delta V = 0.01V$	1 mark SUBTOTAL 1
4(b)(i)	$E = \frac{P}{D}.\frac{dD}{dP}$ $= \frac{P}{D}.\frac{\Delta D}{\Delta P}$ $= \frac{\Delta D}{D}.\frac{P}{\Delta P}$ $= \frac{\Delta D}{D}\times 100 \; as \; \frac{\Delta P}{P} = 0.01$ $= percentage \;\; change \;\; in \;\; demand$	1 mark 1 mark 1 mark SUBTOTAL 3
4(b)(ii)	E is usually negative because an increase in price will usually result in a decreasing demand	1 mark SUBTOTAL 1
4(b)(iii)1	$E = \frac{P}{D}.\frac{dD}{dP}$ $= \frac{P}{D}.-4$ $= \frac{D-200}{4D}.-4$ $= \frac{200-D}{D}$ or $\frac{P}{P-50}$	1 mark 1 mark SUBTOTAL 2
4(b)(iii)2	$E = -1$ If $-1 = \frac{p}{p-50}$ $p = 25$	1 mark SUBTOTAL 1
4(b)(iii)3	$0 \le P < 50$ ensures E is negative and E is not defined for $P = 50$	1 mark SUBTOTAL 1
4(b)(iv)1	$R = PD$ $\frac{dR}{dP} = D + P\frac{dD}{dP}$ $= D + P\frac{DE}{P}$ $= D(E+1)$	1 mark 1 mark SUBTOTAL 2
4(b)(iv)2	For maximum revenue $\frac{dR}{dP} = 0$ $D(E+1) = 0$ $D = 0 \; or \; E = -1$ There is no revenue for D=0 Hence $\lvert E \rvert = 1$	1 mark 1 mark SUBTOTAL 2
4(b)(iv)3	$\frac{dR}{dD} = P + D\frac{dP}{dD}$ $= P + D.\frac{P}{D}.\frac{1}{E}$ $= \frac{P}{E}(E+1)$	1 mark 1 mark SUBTOTAL 2

4(b)(v)1	As $P > 0$ we must have $a - bD^2 > 0$ ie $D < \sqrt{\frac{a}{b}}$	1 mark 1 mark SUBTOTAL 2
4(b)(v)2	$\frac{dR}{dP} = D(E+1)$ $E = \frac{P}{D} \cdot \frac{dD}{dP}$ $P = a - bD^2$ $1 = -b2D\frac{dD}{dP}$ $\frac{dD}{dP} = -\frac{1}{2bD}$ $E = -\frac{P}{2bD^2}$ $\frac{dR}{dP} = D\left(1 - \frac{P}{2bD^2}\right)$ $= D\left(1 - \frac{a - bD^2}{2bD^2}\right)$ $= D\left(\frac{3bD^2 - a}{2bD^2}\right)$ $3bD^2 - a > 0$ $D > \sqrt{\frac{a}{3b}}$ So $\sqrt{\frac{a}{3b}} < D < \sqrt{\frac{a}{b}}$	1 mark 1 mark 1 mark SUBTOTAL 3

5	Note: Do not penalise if the +c term is missing anywhere	
5(a)(i)1	 $\sin^2 x + \cos^2 x = 1$ by Pythagoras	1 mark for unit circle (needed for angles >90) 1 mark for explanation involving Pythagoras SUBTOTAL 2
5(a)(i)2	$\sin^2 x + \cos^2 x = 1$ $\dfrac{\sin^2 x}{\cos^2 x} + 1 = \dfrac{1}{\cos^2 x}$ $\tan^2 x + 1 = \sec^2 x$	1 mark SUBTOTAL 1
5(a)(i)1	$I_n = \int \tan^n x dx$ $= \int \tan^{n-2} x \tan^2 x dx$ $= \int \tan^{n-2} x(\sec^2 - 1)dx$ $= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$ $= \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}$	1 mark 1 mark SUBTOTAL 2
5(a)(i)2	$I_8 = \int \tan^8 x dx$ $= \dfrac{\tan^7 x}{7} - I_6$ $= \dfrac{\tan^7 x}{7} - \left[\dfrac{\tan^5 x}{5} - I_4\right]$ $= \dfrac{\tan^7 x}{7} - \dfrac{\tan^5 x}{5} + I_4$ $= \dfrac{\tan^7 x}{7} - \dfrac{\tan^5 x}{5} + \dfrac{\tan^3 x}{3} - I_2$ $= \dfrac{\tan^7 x}{7} - \dfrac{\tan^5 x}{5} + \dfrac{\tan^3 x}{3} - \left[\dfrac{\tan^1 x}{1} - I_0\right]$ $= \dfrac{\tan^7 x}{7} - \dfrac{\tan^5 x}{5} + \dfrac{\tan^3 x}{3} - \tan x + I_0$ $= \dfrac{\tan^7 x}{7} - \dfrac{\tan^5 x}{5} + \dfrac{\tan^3 x}{3} - \tan x + x + c$	1 mark for starting correctly 1 mark for correct intermediate steps 1 mark for I_0 SUBTOTAL 3

5(b)(i)	$I = \int \cos^5 x dx$ $= \int (\cos^2 x)^2 \cos x dx$ $= \int (1 - \sin^2 x)^2 \cos x dx$ Let $t = \sin x$ $dt = \cos x dx$ $I = \int (1-t^2)^2 dt$ $= \int (1 - 2t^2 + t^4) dt$ $= t - \frac{2}{3}t^3 + \frac{1}{5}t^5$ $= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$	1 mark 1 mark 1 mark SUBTOTAL 3
5(b)(ii)1	$f(x) = \cos^5 x \Rightarrow f(0) = 1$ $f'(x) = -5\cos^4 x \sin x \Rightarrow f'(0) = 0$ $f''(x) = -5\cos^5 x + 20\cos^3 x \sin^2 x \Rightarrow f''(0) = -5$	1 mark 1 mark SUBTOTAL 2
5(b)(ii)2	$a = f(0) = 1$ $b = f'(0) = 0$ $2c = f''(0) \Rightarrow c = -2.5$	1 mark SUBTOTAL 1
5(b)(ii)3	$\int \cos^5 x dx \approx \int (1 - 2.5x^2) dx$ $= x - \frac{2.5}{3}x^3 + c$	1 mark SUBTOTAL 1
5(c)(i)	$\cos^5 0.7 = 0.262$ $1 - 2.5 \times 0.8^2 = -0.6$	1 mark for both correct SUBTOTAL 1
5(c)(ii)1	$\int_0^{0.6} \cos^5 x dx = \left[\sin x + \frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x\right]_0^{0.6} = 0.456$	1 mark SUBTOTAL 1
5(c)(ii)2	$\left[x - \frac{2.5}{3}x^3\right]_0^{0.6} = 0.42$	1 mark SUBTOTAL 1
5(c)(ii)3	Using Simpson's Rule $\frac{0.1}{3}[1 + 4\times0.975 + 2\times0.904 + 4\times0.796 + 2\times0.663 + 4\times0.521 +$ $2\times0.383 + 4\times0.262 + 0.164]$ $= 0.456$	1 mark 1 mark SUBTOTAL 2

6(a)(i)	$\frac{d}{dt}\left(e^{\beta t}w\right)=w\frac{d}{dt}e^{\beta t}+e^{\beta t}\frac{dw}{dt}$ $=\beta we^{\beta t}+e^{\beta t}\frac{dw}{dt}$	1 mark 1mark SUBTOTAL 2
6(a)(ii)	$\frac{dw}{dt}=\alpha-\beta w$ $e^{\beta t}\frac{dw}{dt}=\alpha e^{\beta t}-\beta we^{\beta t}$ $\alpha e^{\beta t}=e^{\beta t}\frac{dw}{dt}+\beta we^{\beta t}$ $\frac{d}{dt}\left(we^{\beta t}\right)=\alpha e^{\beta t}$ $we^{\beta t}=\frac{\alpha}{\beta}e^{\beta t}+c$ $w=\frac{\alpha}{\beta}+ce^{-\beta t}$	1 mark 1 mark 1 mark 1 mark SUBTOTAL 4
6(b)(i)	Setting $m=w^3$ $\frac{dm}{dt}=3w^2\frac{dw}{dt}$ $3w^2\frac{dw}{dt}=Aw^2-Bw^3$ $\frac{dw}{dt}=\frac{A}{3}-\frac{B}{3}w$ minus	1 mark 2 marks (1 for each side) 1 mark SUBTOTAL 4
6(b)(ii)	$w=\frac{A}{B}+ce^{-\frac{B}{3}t}$ on substituting $\alpha=\frac{A}{3}$ $\beta=\frac{B}{3}$ $m=\left(\frac{A}{B}+ce^{-\frac{B}{3}t}\right)^3$	1 mark 1 mark SUBTOTAL 2
6(b)(iii)	Also, when $t=0$, $m=0$ so $c=-\frac{A}{B}$ Thus $m=\left(\frac{A}{B}-\frac{A}{B}e^{-\frac{B}{3}t}\right)^3$ $=\left(\frac{A}{B}\right)^3\left(1-e^{-\frac{B}{3}t}\right)^3$ As $t\to\infty$ $m\to\left(\frac{A}{B}\right)^3$ so $M=\left(\frac{A}{B}\right)^3$ so $m=M\left(1-e^{-\frac{B}{3}t}\right)^3$	1 mark 1 mark 1 mark SUBTOTAL 3
6(b)(iv)1	It goes through the origin because it starts with mass 0 at time 0	SUBTOTAL 1
6(b)(iv)2	As t increases $e^{-\frac{B}{3}t}$ decreases Hence $1-e^{-\frac{B}{3}t}$ increases	1 mark 1mark SUBTOTAL 2
6(b)(iv)3	$m=M$	2 marks SUBTOTAL 2