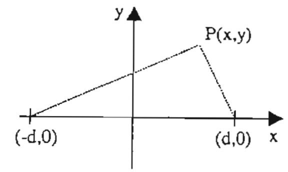
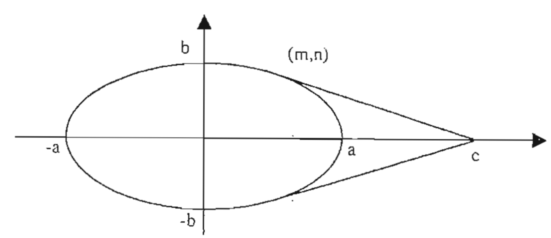
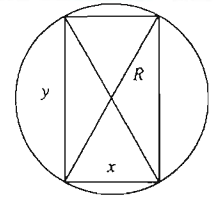
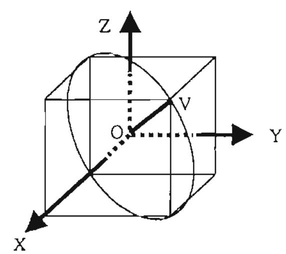
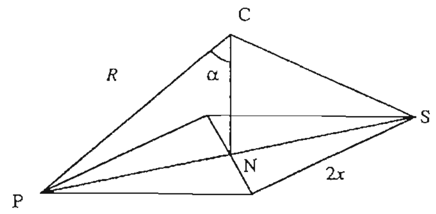
NZEST SCHOLARSHIP EXAMINATION 2000: MATHEMATICS WITH CALCULUS ANSWERS

1ai	$1.75 \to 5.7546 \quad 3.5 \to 33.1155 \quad 5 \to 146.4132$		1	=1
1aii	$\int_1^5 e^x dx = \left[e^x\right]_1^5 = 145.7$		1	=1
1aiii	$\int_1^5 e^x dx = \frac{1}{2}\left[e^1 + e^5 + 2(e^2 + e^3 + e^4\right]$ $= \frac{1}{2}\left[2.72 + 148.41 + 2(7.39 + 20.09 + 54.60\right]$ $= 157.6$		1 1	=2
1bi	$f(1.5) + f(2.5) + f(3.5) + f(4.5) = 4.48 + 12.18 + 33.12 + 90.02$ $= 139.8$		1 1	=2
1bii	$\frac{1}{2}\left[f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + + f(3.75) + f(4.25) + f(4.75)\right]$ $= \frac{1}{2}\left[3.49 + 5.75 + 9.49 + 15.64 + 25.79 + 42.52 + 70.11 + 115.58\right]$ $= 144.2$		1 1 1	=3
1ci	Number being treated = $300e^{\frac{-15}{20}} = 141.7$	Accept 141 or 142	1	=1
1cii	Length of sub interval = $\frac{15}{n}$		1	=1
1ciii	n intervals, 150 patients so $\frac{150}{n}$ admitted.		1	=1
1civ1	$m_j = \frac{1}{2}\left[\frac{15(j-1)}{n} + \frac{15j}{n}\right]$ $= \frac{15}{2n}\left[j - 1 + j\right]$ $= \frac{15(2j-1)}{2n}$		1 1	=2
1civ2	Time $= 15 - m_j$		1	=1
1cv	fraction = $e^{\frac{-t}{20}} = e^{\frac{m_j - 15}{20}}$		1	=1
1cvi	At the end of 15 months the number of patients still being treated is the number still left from the original 300 plus the sum of the numbers of patients still being treated who were admitted in the subintervals ie $300 \times e^{\frac{-15}{20}} + \sum_{j=0}^{n-1} 10e^{\frac{m_j - 15}{20}} \frac{15}{m}$	1 for the explanation	1	=1
1cvii	As $n \to \infty$ $300e^{-0.75} = 141.71$ is constant $\sum_{j=0}^{n-1} 10e^{\frac{m_j-15}{20}} \frac{15}{m} \to \int_0^{15} 10e^{\frac{x-15}{20}} dx$ $= 10e^{\frac{-15}{20}} \int_0^{15} e^{\frac{x}{20}} dx$ $= 200e^{-0.75}\left[e^{\frac{x}{20}}\right]_0^{15}$ $= 105.5$ So number is 247	Accept 248	1 1 1	=3

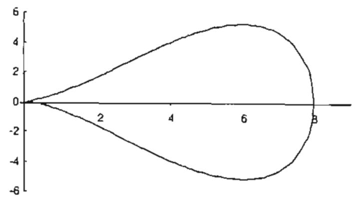
2ai	 Dist = $\sqrt{(x-d)^2 + y^2}$		1	=1
2aii	$\sqrt{(x-d)^2 + y^2} + \sqrt{(x+d)^2 + y^2} = 2R$		1	=1
2aiii	When it crosses the line joining the planets $y = 0$ $\sqrt{(x-d)^2} + \sqrt{(x+d)^2} = 2R$ ie $(x-d) + (x+d) = 2R$ so $x = R$ or $-(x-d) - (x+d) = 2R$ so $x = -R$		1 1 1	=3
2aiv	When the comet is equidistant P must lie on the perpendicular bisector of (-d,0) and (d,0). The diagram shows this means $x = 0$. $\sqrt{d^2 + y^2} + \sqrt{d^2 + y^2} = 2R$ $\Rightarrow 2\sqrt{d^2 + y^2} = 2R$ $\Rightarrow \sqrt{d^2 + y^2} = R$ $\Rightarrow d^2 + y^2 = R^2$ $\Rightarrow y^2 = R^2 - d^2$		1 1	=2
2av	$\sqrt{(x-d)^2 + y^2} + \sqrt{(x+d)^2 + y^2} = 2R$ $\therefore \sqrt{(x-d)^2 + y^2} = 2R - \sqrt{(x+d)^2 + y^2}$ $\therefore (x-d)^2 + y^2 = 4R^2 + (x+d)^2 + y^2 - 4R\sqrt{(x+d)^2 + y^2}$ $\therefore x^2 - 2dx + d^2 + y^2 = 4R^2 + x^2 + 2dx + d^2 + y^2 - 4R\sqrt{(x+d)^2 + y^2}$ $\therefore 4R\sqrt{(x+d)^2 + y^2} = 4R^2 + 4dx$ $\therefore R\sqrt{(x+d)^2 + y^2} = R^2 + dx$ $\therefore R^2\left((x+d)^2 + y^2\right) = R^4 + 2R^2dx + d^2x^2$ $\therefore R^2x^2 + 2R^2dx + R^2d^2 + R^2y^2 = R^4 + 2R^2dx + d^2x^2$ $\therefore (R^2 - d^2)x^2 + R^2y^2 = R^2(R^2 - d^2)$ $\therefore \frac{x^2}{R^2} + \frac{y^2}{R^2 - d^2} = 1$		1 1 1 1	=4
2bi		(m,n) not necessary	1	=1

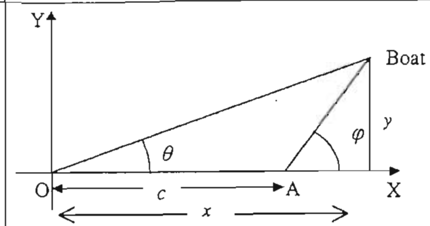
2bii	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\therefore \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$ $\therefore \frac{dy}{dx}=\frac{-b^2x}{a^2y}$ Hence $\therefore \frac{dy}{dx}=\frac{-b^2m}{a^2n}$ at (m,n)		1 1 =2
2biii	The gradient of the line joining (m,n) to $(c,0)$ is $\frac{n}{m-c}$ $\therefore \frac{n}{m-c}=\frac{-b^2m}{a^2n}$ $\therefore a^2n^2=-b^2m^2+b^2mc$ $\therefore a^2n^2+b^2m^2=b^2mc$ $\therefore \frac{n^2}{b^2}+\frac{m^2}{a^2}=\frac{mc}{a^2}$ $\therefore \frac{mc}{a^2}=1$ Since (m,n) lies on the ellipse $\frac{m^2}{a^2}+\frac{n^2}{b^2}=1$ $\therefore m=\frac{a^2}{c}$		1 1 1 =3
2biv	$\frac{m^2}{a^2}+\frac{n^2}{b^2}=1$ $\therefore \frac{1}{a^2}\frac{a^4}{c^2}+\frac{n^2}{b^2}=1$ $\therefore \frac{n^2}{b^2}=1-\frac{a^2}{c^2}$ $\therefore n^2=\frac{b^2}{c^2}\left(c^2-a^2\right)$ $\therefore n=\frac{b}{c}\sqrt{c^2-a^2}$		1 1 =2
2bv	As $c\to a$ $n\to 0$ and $m\to a$		1 =1

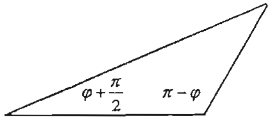
3a	 Let R be the radius of the circle x, y be the lengths of the two sides Then $x^2+y^2=R^2$ and the area is $A=xy$ $=x\sqrt{4R^2-x^2}$ $\frac{dA}{dx}=\sqrt{4R^2-x^2}+x\frac{1}{2\sqrt{4R^2-x^2}}.-2x$ $=\sqrt{4R^2-x^2}-\frac{x^2}{\sqrt{4R^2-x^2}}$ $=\frac{4R^2-2x^2}{\sqrt{4R^2-x^2}}$ $\frac{dA}{dx}=0$ implies $4R^2-2x^2=0$ $\therefore x^2=2R^2$ or $x=\sqrt{2}R$ and $x^2+y^2=R^2 \Rightarrow y=\sqrt{2}R$ Hence $x=y$ so we have a square. To show this is a maximum look at the sign of $\frac{dy}{dx}$. $\frac{dy}{dx}>0$ $4R^2-2x^2>0$ when $-2x^2>-4R^2$ ie $x^2<2R^2$ ie $x<\sqrt{2}R$ Similarly $\frac{dy}{dx}<0$ when $x>\sqrt{2}R$ So gradient goes from +ve when $x<\sqrt{2}R$ to −ve when $x>\sqrt{2}R$ ie we have a maximum.		1 1 1 1 1 1 1 1 1 1 =10
b	The diagram shows one of the half diagonals of the cube which goes from the centre O to a vertex V. OV = R is the radius of the sphere. Then, if the coordinates of V are (x, y, z) $x^2+y^2+z^2=R^2$ But $x=y=z$ since we have a cube. ie $3x^2=R^2$ or $x=\frac{R}{\sqrt{3}}$ So sides are each $\frac{2R}{\sqrt{3}}$ And Volume $=\left(\frac{2R}{\sqrt{3}}\right)^3=\frac{8R^3}{3\sqrt{3}}$ 		1 1 1 1 1 =5

3c				
	$PS^2 = (2x)^2 + (2x)^2$ $= 8x^2$ $= 8\times\left(\frac{R}{\sqrt{3}}\right)^2$ $= \frac{8}{3}R^2$		1	
	So $PS = \frac{2\sqrt{2}}{\sqrt{3}}R$ Or $PN = \sqrt{\frac{2}{3}}R$		1	
	also $CN^2 = R^2 - PN^2$ $= R^2 - \frac{2}{3}R^2$ $= \frac{R^2}{3}$ $CN = \frac{R}{\sqrt{3}}$		1	
	$\tan\alpha = \frac{PN}{CN} = \sqrt{\frac{2}{3}}R\times\frac{\sqrt{3}}{R} = \sqrt{2}$		1	
	What is required is $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\sqrt{2}}{1-2} = -2\sqrt{2}$		1	=5

4ai	$\int 8y\,dy = \int x^2(6-x)dx$		1	
	$\therefore 4y^2 = 2x^3 - \frac{x^4}{4} + c$			
	$\therefore 16y^2 = 8x^3 - x^4 + 4c$		1	
	$y(0) = 0 \Rightarrow 0 = 0 - 0 + 4c \Rightarrow c = 0$		1	
	Equation is $16y^2 = x^3(8-x)$			=3
4aii	If (x, y) satisfies the equation, so does $(x, -y)$		1	=1
4aiii	$x = 4(1+\sin\theta)$ $y = 4x\cos\theta$			
	$\sin\theta = \frac{x}{4} - 1$ $\cos\theta = \frac{y}{4x}$		1	
	$\left(\frac{x}{4}-1\right)^2 + \frac{y^2}{16x^2} = 1$		1	
	$\therefore \frac{x^2}{16} - \frac{x}{2} + 1 + \frac{y^2}{16x^2} = 1$ $\therefore x^4 - 8x^3 + y^2 = 0$ $\therefore y^2 = 8x^3 - x^4$ $\therefore y^2 = x^3(8-x)$		1	=3
4aiv	$\frac{dx}{d\theta} = 4\cos\theta$		1	
	$\frac{dy}{d\theta} = 4\cos\theta\cos\theta + 4(1+\sin\theta).-\sin\theta$		1	
	$= 4(\cos^2\theta - \sin\theta - \sin^2\theta)$ $= 4(\cos 2\theta - \sin\theta)$		1	
	$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{\cos 2\theta - \sin\theta}{\cos\theta}$		1	=4
4av	When $\theta = \frac{\pi}{6}$ $\cos\frac{\pi}{3} - \sin\frac{\pi}{6}$ $= \frac{1}{2} - \frac{1}{2}$ $= 0$		1	=1
4avi	$x = 4\left(1+\sin\frac{\pi}{6}\right) = 4\left(1+\frac{1}{2}\right) = 6$		1	
	$y = 4\left(1+\cos\frac{\pi}{6}\right)\sin\frac{\pi}{6} = 4\left(1+\frac{\sqrt{3}}{2}\right)\frac{1}{2} = 2+\sqrt{3} \approx 3.73 = 4$		1	=2
4avii	Since $16y^2 = x^3(8-x)$, $y = 0$ at $x = 8$ So $\frac{dy}{dx} = \frac{x^2(6-x)}{8y}$ is undefined at $y = 0$			=1

4aviii		1 mark for turning point 1 for perpendicular at 8 1 mark for general shape	3	=3
4b	$V=\pi\int_0^8 y^2$ $=\frac{\pi}{16}\int_0^8 x^3(8-x)$ $=\frac{\pi}{16}\left[\frac{8x^4}{4}-\frac{x^5}{5}\right]_0^8$ $=\frac{\pi}{16}\left(2\times8^4-\frac{8^5}{5}\right)$ $=321.7$		1 1	=2

5a	$\cot(180-\theta)=\frac{\cos(180-\theta)}{\sin(180-\theta)}$ $=\frac{-\cos\theta}{\sin\theta}$ $=-\cot\theta$		1	=1
5bi1	 $y=x\tan\theta$ $y=(x-c)\tan\varphi$ $\therefore x\tan\theta=x\tan\varphi-c\tan\varphi$ $\therefore x(\tan\theta-\tan\varphi)=-c\tan\varphi$ $or\ x=\frac{c\tan\varphi}{\tan\theta-\tan\varphi}$		1 1 1 1	=4
5bi2	$\tan\theta-\tan\varphi=\frac{\sin\theta}{\cos\theta}-\frac{\cos\varphi}{\sin\varphi}$ $=\frac{\sin\theta\sin\varphi-\cos\theta\cos\varphi}{\sin\varphi\cos\varphi}$ $=\frac{\sin(\theta-\varphi)}{\sin\varphi\cos\varphi}$ Hence $x=\frac{c\sin\varphi\cos\theta}{\sin(\theta-\varphi)}$ $y=x\tan\theta$ $=\frac{c\sin\varphi\cos\theta}{\sin(\theta-\varphi)}\times\frac{\sin\theta}{\cos\theta}$ $=\frac{c\sin\varphi\sin\theta}{\sin(\theta-\varphi)}$		1 1 1 1	=4
5bi3	$D^2=x^2+y^2$ $=\left(\frac{c\sin\varphi\cos\theta}{\sin(\theta-\varphi)}\right)^2+\left(\frac{c\sin\varphi\sin\theta}{\sin(\theta-\varphi)}\right)^2$ $=\frac{c^2\sin^2\varphi}{\sin^2(\theta-\varphi)}\left(\cos^2\theta+\sin^2\theta\right)$ $=\frac{c^2\sin^2\varphi}{\sin^2(\theta-\varphi)}$ $D=\frac{c\sin\varphi}{\sin(\theta-\varphi)}$		1 1	=2
5bii1	$\frac{dD}{d\theta}=\frac{-c\sin\varphi}{\sin^2(\theta-\varphi)}\cos(\theta-\varphi)$ $\frac{dD}{d\theta}=0$ when $\cos(\theta-\varphi)=0$ ie $\theta-\varphi=\frac{\pi}{2}$ or $\theta=\varphi+\frac{\pi}{2}$		1 1	=2

5bii2	Drawing out a general triangle (not to scale) to see how the angles lie at a minimum we get:  This means the angle at the top of the triangle is a right angle. (and hence the base angles of the triangle should both be acute) Hence the minimum positions lie on a circle.	1 for angles 1 for locus	1 1	 =2
5ci	$L_{n-1} = \lim_{\theta \to 0} \frac{\sin(n-1)\theta}{\sin\theta}$		1	=1
5cii	$L_n = \lim_{\theta \to 0} \frac{\sin((n-1)\theta + \theta)}{\sin\theta}$ $= \lim_{\theta \to 0} \frac{\sin(n-1)\theta\cos\theta + \cos(n-1)\theta\sin\theta}{\sin\theta}$ $= \lim_{\theta \to 0} \left(\frac{\sin(n-1)\theta}{\sin\theta}\cos\theta + \cos(n-1)\theta \right)$ $\to L_{n-1} + 1 \quad as\, \theta \to 0$		1 1	 =2
5ciii	$L_n = L_{n-1} + 1$ $= L_{n-2} + 1 + 1$ $= L_{n-3} + 1 + 2$ .... $= L_1 + n - 1$ $= n$	1 for expanding to at least n-2 1 for n	1 1	 =2

6ai	Either $z = i$ or $z^2 - 2z + 4 = 0$ In the second case $z = \frac{2 \pm \sqrt{4-16}}{2}$ $= 1 \pm \sqrt{3}i$ Roots are $i,\ 1 \pm \sqrt{3}i$		1 1	 =2
6aii	If the roots are $a, \bar{a}$ then the polynomial has factors $z-a, z-\bar{a}$ Hence a factor $(z-a)(z-\bar{a}) = z^2 - (a+\bar{a})z + a.\hat{a}$ $= z^2 - 2\,\mathrm{Re}(a)z + \|a\|^2$		1 1 1	 =3
6bi	$z = cis\theta$ $(cis\theta)^n + (cis\theta)^{-n} = cis(n\theta) + cis(-n\theta)$ $= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ $= 2\cos n\theta$		1 1	 =2
6bii	$z^n - z^{-n} = 2i\sin n\theta$		1	=1
6ci	$z^{2n} = cis2k\pi$ $z = cis\frac{k\pi}{n} \quad k = -(n-1), -(n-2), \ldots\ldots, -2, -1, 0, 1, 2, \ldots\ldots n-2, n$ (using the fact that any 2n consecutive values of k will do) $z = \mathrm{cis}0 \quad z = \mathrm{cis}\pi \quad z = cis\frac{\pm k\pi}{n} \quad k = 1, \ldots\ldots, n-1$ $z = 1,\ -1,\ \cos\frac{k\pi}{n} \pm \sin\frac{k\pi}{n} \quad k = 0, 1, \ldots\ldots, n-1$		1 1 1	 =3
6cii	As 1 and −1 are roots, the expression has factors $(z-1), (z+1)$ ie a factor of $z^2 - 1$		1	=1
6ciii	Since the roots come in conjugate pairs $z = cis\frac{k\pi}{n} = \cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}$ and $\bar{z} = cis\frac{-k\pi}{n} = \cos\frac{k\pi}{n} - i\sin\frac{k\pi}{n}$ the expression has factors $\left(z - \left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)\right)\left(z - \left(\cos\frac{k\pi}{n} - i\sin\frac{k\pi}{n}\right)\right)$ using the result in aii this gives a factor of $z^2 - 2z\cos\frac{k\pi}{n} + 1$ Hence the expression can be written as $z^{2n} - 1 = (z^2 - 1)\prod_{k=1}^{n-1}\left(z^2 - 2z\cos\frac{k\pi}{n} + 1\right)$ Dividing by z^n gives $z^n - z^{-n} = (z - z^{-1})\prod_{k=1}^{n-1}\left(z - 2\cos\frac{k\pi}{n} + z^{-1}\right)$		1 1 1	 =3
6civ	$2i\sin n\theta = z^n - z^{-n}$ $= (z - z^{-1})\prod_{k=1}^{n-1}\left(z + z^{-1} - 2\cos\frac{k\pi}{n}\right)$ $= 2i\sin\theta\prod_{k=1}^{n-1}\left(2\cos\theta - 2\cos\frac{k\pi}{n}\right)$ So $\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\prod_{k=1}^{n-1}\left(\cos\theta - \cos\frac{k\pi}{n}\right)$		1 1	 =2

6cv	Taking limits of each side gives $n = 2^{n-1}\prod_{k=1}^{n-1}\left(1-\cos\frac{k\pi}{n}\right)$ Using $1-\cos 2A = 2\sin^2\frac{A}{2}$ $n = 2^{n-1}\prod_{k=1}^{n-1} 2\sin^2\frac{k\pi}{2n}$ $= 2^{n-1}\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{2n}\sin\frac{(n-k)\pi}{2n}$ $= 2^{n-1}\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{2n}\cos\frac{k\pi}{2n}$ $= 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}$		1 1 1 =3