NEW ZEALAND QUALIFICATIONS AUTHORITY
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Scholarship, 2004

Calculus

National Statistics

Assessment Report

Assessment Schedule

Calculus, Scholarship, 2004
General Comments

The paper proved to be of a very challenging standard, as might be expected of a scholarship paper.

In order to gain Scholarship with Outstanding Performance, candidates had to demonstrate a very high level of conceptual thinking coupled with excellent problem solving skills, and the ability to sustain algebraic reasoning throughout an extended argument. It was necessary to apply these abilities to questions that were set in unfamiliar contexts, or contained unfamiliar ideas. These candidates also demonstrated the ability to combine thinking from more than one area of mathematics and to produce novel approaches and solutions to the problems.

Candidates achieving scholarship were able to combine a clear conceptual grasp of an abstract problem with a good knowledge of mathematical procedures and strong algebraic skills. They could also apply their knowledge to questions involving unfamiliar contexts, such as Questions 1, 4 and 5. The most common combination of questions to achieve the standard was 2(a)(i), 3(a)(i), 4(a)(i) and 4(a)(ii), combined with one or more other parts, occasionally an SOP part used as replacement evidence. The level of success with both parts of question 4(a) was unexpectedly high. Candidates sometimes made a minor error in their working, but this was ignored if the rest of the solution was correct. Conceptual ideas that these candidates understood well included the use of the principle of the Intermediate Value Theorem in Q3(a)(i) and the role of the constant of integration in Q6(a).

Candidates who did not achieve the scholarship standard often did not understand the concepts involved, failed to complete question parts, did not answer the question as stated, or were unable to apply their knowledge in unfamiliar contexts. Sometimes they spent too much of their time and energy on lengthy algebraic manipulation, but lacked the insight to discern that the algebra was not leading in the right direction. This was particularly true where the answer was given in the question, such as in Q2(a)(i) and 2(b). Often the cause was a failure to simplify algebraic expressions before working with them, a failure to cancel out common factors early on, or introducing extra variables rather than eliminating them.

Unfortunately, there was a large number of candidates who seemed unprepared for work at this level. They were unable to differentiate basic functions such as $f(x) = x^n$ or wrote statements such as $\int f(x)\,dx = \frac{f(x)^{n+1}}{n+1}$, $\frac{x^2}{1+x^2} = \frac{x^2}{1} + \frac{x^2}{x^2}$ or $x^2(x^2+2) = 4x - 1$ so $x^2 = 4x - 1$ or $x^2 + 2 = 4x - 1$, or used the quadratic formula to solve $-5x^2 + 2Lx = 0$.

Calculus Scholarship (93202)
National Statistics

Number of Results	Percentage achieved		
	Not Achieved	Scholarship	Outstanding
972	74.8	22.6	2.6

Assessment Report

Question 1

(a) This was a relatively straightforward optimisation problem; while the majority of candidates understood the principles of finding an expression for V, differentiating it, solving $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$, they found a formula for the height of the pyramid difficult to derive. A failure to simplify a correct height of $\sqrt{(L-x)^2 - x^2}$ often led to unnecessarily complicated differentiation and algebraic manipulation. A noticeable number of candidates overgeneralised, considering the volume of any solid to be the integral of the surface area – a cube would be a good counter-example for them to consider.

(b) This proved very difficult, with only better candidates getting the required answer. The chain rule principle was recognised as important, but the application of ratios was not well understood.

Question 2

(a) (i) This question was well answered by a good number of candidates, although some did not take anything like a direct route to the answer, with solutions often extended due to the presence of a printed answer. An inability to differentiate implicitly caused serious problems for some candidates. Other candidates stopped at a version of $2b^2x = ay^2$, but lacked the insight to substitute $b^2 = 2a^2$, or $y^2 = 4ax$ to get the given result. Sometimes they proceeded by replacing x as a function of y at the same time as replacing y as a function of x. Occasionally candidates made non-minor errors in their working, such as incorrect cancelling of the *ys* in $\frac{2a}{y} = \frac{a^2y}{b^2x}$.

(a) (ii) This was one of the easier SOP problems. It was pleasing to see the handling of the surds involved and the correct solution obtained by a good number of candidates, although some found $x = -a \pm a\sqrt{2}$ or $a(\sqrt{2} - 1)$ but did not progress to find OP. A number of candidates solved $x^2 + 2x - a^2 = 0$ incorrectly, obtaining $x = a$ for P. Some candidates worked correctly in decimals.

(b) Most students avoided parameters and worked well with the surds, with really good algebra shown by quite a few to get to the correct final answer. A number of candidates failed to see any difference between the general (x,y) and (X_p, y_p), and so, for example, failed to substitute in the gradients such as $\frac{dy}{dx} = \frac{2a}{y_P}$. It proved much harder to get the correct answer working in decimals here.

Question 3

(a) (i) This was well done, by a variety of methods. Often the principles of continuity and the Intermediate Value Theorem were well used. Students were clearly able to think about the principles involved rather than just running through a learned procedure. Decimals were widely used successfully.

(a) (ii) This was well done by the better candidates, although some made numerical errors towards the end of the question. Most quickly spotted the required translation parallel to the x-axis, and good use of modulus sign in $\int \frac{1}{u}du = \ln|u| + C$ was noticed. Some confused the volume of rotation about the x-axis with the correct one about the y-axis. Allowance was made for those candidates who took the rotational volume away from the cylinder, to obtain the volume of material making up their bowl.

(b) (i) A conceptual problem that was one of the more challenging in the paper, and proved beyond most students. A number attempted to use examples, making $f(x)$ some arbitrary function (often x^2). Those who did use a version of $\int_0^a f(a-x)dx = [-F(a-x)]_0^a$ often missed the negative sign and then had to arrange for the result to appear.

(b) (ii) This was a very difficult question, with few leads given beyond the use of part (i). Candidates often stopped after applying this result. Success was a real accomplishment for those few who managed to solve it. A number of good candidates erroneously thought that $\int_0^a f(x)dx = \int_0^a g(x)dx \Leftrightarrow f(x) = g(x)$.

Question 4

(a) (i) This was very well done and was one of the most successfully attempted questions in the paper. Most unsuccessful candidates failed because they appeared not to not know that for $z = x + \mathrm{i}y$,
$|\, z \,| = \sqrt{x^2 + y^2}$, and instead kept the i's in their attempt to calculate the moduli.

(a) (ii) This was very well done and was one of the most successfully attempted questions on the paper.

(b) This was beyond the reach of all but a very few candidates. Many got the roots of the quartic, but were then unable to form the product of two quadratic expressions using their roots. A number worked in surds or decimals as far as a product of two quadratic expressions, and then found the final connection impossible to make. Some credit was given for this.

Question 5

(a) Not as well done as expected. Many candidates correctly expressed x and y as functions of r and θ, but when differentiating forgot that r was itself a function of θ and so the product rule was necessary. A sizeable number put $x = r$ and $y = \theta$ and so failed to make any progress.

(b) This was successfully completed only by the very best students. Several very innovative methods were seen resulting in equivalent versions of the expected answer. Often the use of the exterior angle, and the sum of the angles equal to π, were missed. It seems that geometric thinking was relatively weak.

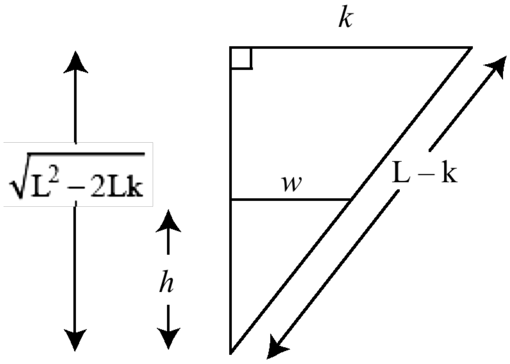
Question 6

(a) This was done well by the good candidates, but too many ignored the value of the integration constants, either missing them out altogether or leaving them unevaluated in the final answer. Prior knowledge of the co-ordinates of the maximum/highest point did not appear to assist candidates, rather the opposite in fact.

(b) (i) and (ii) These were not well handled at all and only the very best candidates were able to make any progress beyond substituting for x and y.

Scholarship Calculus 2004 (93202)
Assessment Schedule

No.	Evidence	Code	Judgement
1(a)	Slant height of the pyramid $= \dfrac{2L - 2x}{2} = L - x$ By Pythagoras' theorem $$(\text{height of pyramid})^2 = (L - x)^2 - x^2 = L^2 - 2Lx + x^2 - x^2 = L^2 - 2Lx$$ $$\text{height of pyramid} = \sqrt{L^2 - 2Lx}$$ $$\text{Volume} = \frac{1}{3}(2x)^2 \times \sqrt{L^2 - 2Lx} = \frac{1}{3}4x^2 \times \sqrt{L^2 - 2Lx} \quad \text{cm}^3$$ Hence the value of x to give max. volume: $$\frac{dV}{dx} = \frac{4}{3}2x\sqrt{L^2 - 2Lx} + \frac{4}{3}x^2\frac{1}{2}\left(L^2 - 2Lx\right)^{\frac{-1}{2}}(-2L)$$ $$= \frac{4x}{3}\left(L^2 - 2Lx\right)^{\frac{-1}{2}}\left[2\left(L^2 - 2Lx\right) - Lx\right]$$ $$= \frac{4x}{3}\left(L^2 - 2Lx\right)^{\frac{-1}{2}}\left[2L^2 - 5Lx\right] = 0 \text{ for max. \& min.}$$ $x = 0$ or $x = \dfrac{2L^2}{5L} = \dfrac{2L}{5}$ $(L \neq 0)$ So for maximum volume $x = \dfrac{2L}{5}$ cm	S1	Units of measurement not required. Any correct alternative method acceptable (eg use of different triangles). Accept maximising V^2 $x = 0$ not required. Units not required No alternative.

No.	Evidence	Code	Judgement
1(b)	Given $\frac{dV}{dt} = \frac{k^2}{3L}$; to find $\frac{dh}{dt}$  Using similar triangles: $\frac{w}{k} = \frac{h}{\sqrt{L^2 - 2Lk}}$ $w = \frac{kh}{\sqrt{L^2 - 2Lk}}$ $V = \frac{1}{3}(2w)^2 h$ $= \frac{4}{3} \times \frac{k^2 h^3}{L^2 - 2Lk}$ Using the Chain Rule: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\frac{dV}{dt} = \frac{4k^2h^2}{L(L-2k)} \times \frac{dh}{dt}$ $\frac{dh}{dt} = \frac{L(L-2k)}{4k^2h^2} \times \frac{k^2}{3L} = \frac{L-2k}{12h^2}$ Hence, when $\frac{dh}{dt} > \frac{L-2k}{h+1}$ $\frac{L-2k}{12h^2} > \frac{L-2k}{h+1}$ $h + 1 > 12h^2 \quad (h \geq 0,\ L - 2k \neq 0)$ $\therefore 12h^2 - h - 1 < 0$ $(4h+1)(3h-1) < 0$ $\therefore 0 \leq h < \frac{1}{3}$ cm. $\quad (h \geq 0)$	**SOP1** (Possible replacement for SOP2).	Any correct alternative method acceptable. Or equivalent: eg $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ Units of measurement not required. Accept $0 < h < \frac{1}{3}$ or $h < \frac{1}{3}$.

No.	Evidence	Code	Judgement
2(a)(i)	For the parabola: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4a}{2y} = \frac{2a}{y}$ For the ellipse: $\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-b^2 x}{a^2 y}$ At $P(x_1,y_1)$, since the tangents are perpendicular and the product = –1 $\frac{2a}{y_1} \times \frac{-b^2 x_1}{a^2 y_1} = -1$ $2b^2 x_1 = a{y_1}^2$ but (x_1,y_1) lies on $y^2 = 4ax$ (or use parameters) $2b^2 x_1 = a4ax_1$ $b^2 = 2a^2$	**S2** (Possible replacement for S1)	Working with x_1 and y_1 not required. Possible alternative method using parameters is acceptable.

No.	Evidence	Code	Judgement
2(a)(ii)	At the point P the curves intersect: $4ax = b^2\left(1 - \frac{x^2}{a^2}\right)$ $4a^3x = b^2\left(a^2 - x^2\right)$ $b^2x^2 + 4a^3x - a^2b^2 = 0$ but $b^2 = 2a^2$ from part (a) $2a^2x^2 + 4a^3x - 2a^4 = 0$ $x^2 + 2ax - a^2 = 0 \quad (a \neq 0)$ **OR** use $\frac{x^2}{a^2} + \frac{4ax}{2a^2} = 1$ to get the above equation $x = \frac{-2a \pm \sqrt{4a^2 + 4a^2}}{2}$ $x = -a \pm a\sqrt{2} \quad = a(-1 \pm \sqrt{2})$ **OR** use $\left(x + a\right)^2 = 2a^2$ $x + a = \pm a\sqrt{2}, \quad x = \pm a\sqrt{2} - a$ but $x \geq 0$ and since $a > 0$ $x = a(-1 + \sqrt{2})$ Hence $y^2 = 4a^2(-1 + \sqrt{2})$ using the parabola and the distance $OP^2 = x^2 + y^2$ Pythagoras $OP^2 = a^2(-1 + \sqrt{2})^2 + 4a^2(-1 + \sqrt{2})$ $a^2(3 - 2\sqrt{2} - 4 + 4\sqrt{2})$ $= a^2(-1 + 2\sqrt{2})$ and $OP = a\sqrt{2\sqrt{2} - 1}$	**SOP1**	Any correct alternative method accepted (eg from perpendicular gradients). Conditions on x and a not required. Or equivalent. Accept $1.352a$ or equivalent decimal.

No.	Evidence	Code	Judgement
2(b)	Using $(at^2, 2at)$ for the parabola $\frac{dy}{dx} = \frac{1}{t}$ so the tangent to the parabola is: $y - 2at = \frac{1}{t}(x - at^2)$ For M, $y = 0$ so at M $x = -at^2$ and M $= \left(-at^2, 0\right)$ Using $(a\cos\theta, b\sin\theta)$ for the ellipse: $\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$ so the tangent to the ellipse is: $y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ For N, $y = 0$ so at N $x = \frac{a(\sin^2\theta + \cos^2\theta)}{\cos\theta}$ $x = \frac{a}{\cos\theta}$ and N $= (\frac{a}{\cos\theta}, 0)$ Hence MN $= at^2 + \frac{a}{\cos\theta}$ But at the points of intersection (P here) $at^2 = a\cos\theta = a(-1 + \sqrt{2})$ from part (a)(ii) Hence MN $= a(-1+\sqrt{2}) + \frac{a}{(-1+\sqrt{2})}$ $= a(-1+\sqrt{2}) + \frac{a(-1-\sqrt{2})}{(-1+\sqrt{2})(-1-\sqrt{2})}$ $= a(-1 + \sqrt{2} + 1 + \sqrt{2})$ $= 2\sqrt{2}a$ **OR** P $= \left(a(\sqrt{2}-1), 2a\sqrt{\sqrt{2}-1}\right)$ Parabola: at P on the parabola $\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2a\sqrt{\sqrt{2}-1}} = \frac{1}{\sqrt{\sqrt{2}-1}}$ Hence tangent is: $y - 2a\sqrt{\sqrt{2}-1} = \frac{1}{\sqrt{\sqrt{2}-1}}\left(x - a(\sqrt{2}-1)\right)$ when $y = 0$	**SOP3** (Possible replacement for SOP2)	Any correct alternative method accepted.

	$-2a\left(\sqrt{2}-1\right)=x-a(\sqrt{2}-1)$ $x=-a\left(\sqrt{2}-1\right)$ at M Ellipse: at P on the ellipse $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2x}{y}=-\frac{2a\left(\sqrt{2}-1\right)}{2a\sqrt{\sqrt{2}-1}}=-\sqrt{\sqrt{2}-1}$ Hence tangent is: $y-2a\sqrt{\sqrt{2}-1}=-\sqrt{\sqrt{2}-1}\left(x-a\left(\sqrt{2}-1\right)\right)$ when $y=0$ $2a=x-a\left(\sqrt{2}-1\right)$ $x=2a+a\left(\sqrt{2}-1\right)=a\left(1+\sqrt{2}\right)$ at N Hence MN = $a\left(1+\sqrt{2}\right)-\left(-a\left(\sqrt{2}-1\right)\right)=2\sqrt{2}a$. **OR** $\mathrm{P}=\left(a(\sqrt{2}-1),2a\sqrt{\sqrt{2}-1}\right)$ Let Q be the foot of the perpendicular from P to the x-axis. Using gradients: $\frac{\mathrm{PQ}}{\mathrm{MQ}}=\frac{1}{\sqrt{\sqrt{2}-1}}$, but $\mathrm{PQ}=2a\sqrt{\sqrt{2}-1}$ $\mathrm{MQ}=2a\sqrt{\sqrt{2}-1}\times\sqrt{\sqrt{2}-1}=2a\left(\sqrt{2}-1\right)$ Similarly: $\frac{\mathrm{PQ}}{\mathrm{NQ}}=\left\|\frac{-2x}{y}\right\|=\left\|\frac{-2a\left(\sqrt{2}-1\right)}{2a\sqrt{\sqrt{2}-1}}\right\|=\sqrt{\sqrt{2}-1}$, but $\mathrm{PQ}=2a\sqrt{\sqrt{2}-1}$ $\mathrm{NQ}=\frac{2a\sqrt{\sqrt{2}-1}}{\sqrt{\sqrt{2}-1}}=2a$ So MN = MQ + NQ = $2a\left(\sqrt{2}-1\right)+2a=2\sqrt{2}a$.		Accept correct use of absolute value and addition.

No.	Evidence	Code	Judgement
3(a) (i)	$\frac{dy}{dx} = \frac{2x(1+x^2)-2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$ When $\frac{dy}{dx} = \frac{1}{2}$ $\frac{2x}{(1+x^2)^2} = \frac{1}{2}$ $x^4 + 2x^2 - 4x + 1 = 0$ but $x = 1$ is a solution, so by division or otherwise $(x-1)(x^3 + x^2 + 3x - 1) = 0$ and any other solutions are from $x^3 + x^2 + 3x - 1 = 0$ Let $g(x) = x^3 + x^2 + 3x - 1$, then $g\left(\frac{1}{4}\right) = \frac{1+4+48-64}{64} = -\frac{11}{64} < 0$ $g\left(\frac{1}{2}\right) = \frac{1+2+12-8}{8} = \frac{7}{8} > 0$ Hence there is a root in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$ $g(x) = 0$ for some $\frac{1}{4} < x < \frac{1}{2}$. **OR** (while the question does say HENCE, accept the following as an alternative). $f(x) = x^4 + 2x^2 - 4x + 1$ $f\left(\frac{1}{4}\right) = \frac{33}{256} = 0.1289 > 0$ $f\left(\frac{1}{2}\right) = -\frac{7}{16} = -0.4375 < 0$ Hence there is a root in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$ **OR** while the question does say HENCE, accept the following as an alternative.	**S1** (Possible replacement for S2)	Any correct alternative method accepted. Accept $-\frac{11}{64} = -0.1719$, $\frac{7}{8} = 0.875$ or equivalent. Accept $x = 0.2956$ obtained by Newton-Raphson method, graphic calculator, etc. Explicit use of, or mention of intermediate value theorem, or continuity not required.

	$\frac{dy}{dx} = \frac{2x}{(1+x^2)^2}$ $x = \frac{1}{4},\ \frac{dy}{dx} = \frac{\frac{1}{2}}{(1+\frac{1}{16})^2} = \frac{128}{289} < \frac{1}{2}$ and when $x = \frac{1}{2},\ \frac{dy}{dx} = \frac{1}{(1+\frac{1}{4})^2} = \frac{16}{25} > \frac{1}{2}$ note $\frac{128}{289} = 0.443,\quad \frac{16}{25} = 0.64$ or equivalent Hence there is a value of $\frac{dy}{dx} = \frac{1}{2}$ in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$ $\frac{dy}{dx} = \frac{1}{2}$ for some $\frac{1}{4} < x < \frac{1}{2}$		Continuous derivative may be assumed. Accept decimals.

No.	Evidence	Code	Judgement
3(a) (ii)	First translate the function 1 unit in the negative x direction to get $y=\frac{x^2}{1+x^2}$. The volume of revolution is then $V=\int_0^{\frac{1}{2}} \pi x^2 \mathrm{d}y$ and $y=\frac{x^2}{1+x^2}$, so rearrange to get $x^2=\frac{y}{1-y}$ and $V=\pi\int_0^{\frac{1}{2}}\left(\frac{1}{1-y}-1\right)\mathrm{d}y$ $V=\pi\left[-\ln\lvert 1-y\rvert-y\right]_0^{\frac{1}{2}}=\pi(-\ln(\frac{1}{2})-\frac{1}{2})$ $=\pi(\ln(2)-\frac{1}{2})$ units^3. **OR** to integrate using substitution: $V=\int_0^{\frac{1}{2}}\pi\frac{y}{1-y}\mathrm{d}y$ *let* $u=1-y$ $\mathrm{d}u=-\mathrm{d}y$ $y=1-u$ $V=\int_1^{\frac{1}{2}}\pi\frac{1-u}{u}(-\mathrm{d}u)$ $=\int_{\frac{1}{2}}^1\pi\left(\frac{1}{u}-1\right)\mathrm{d}u$ $=\pi\left[\ln\lvert u\rvert-u\right]_{\frac{1}{2}}^1$ $=\pi((0-1)-(\ln(\frac{1}{2})-\frac{1}{2}))$ $=\pi(\ln 2-\frac{1}{2})$ units^3	**S1** (Possible replacement for S2)	Any correct alternative method accepted, including working with no translation. Units are not required. Final answer does not need to be simplified. Accept 0.6068, 0.193π or equivalent. Units not needed.

No.	Evidence	Code	Judgement
3(b) (i)	Consider $I = \int_0^a f(a-x)\mathrm{d}x$. Let $u = a - x$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = -1$, and when $x = 0$, $u = a$; when $x = a$, $u = 0$, so $I = \int_a^0 -f(u)\mathrm{d}u$ $I = \int_0^a f(u)\mathrm{d}u = \int_0^a f(x)\mathrm{d}x$. **OR** Let $F(x)$ be an antiderivative of f $\int_0^a f(x)\mathrm{d}x = \left[F(x)\right]_0^a = F(a) - F(0)$ $\int_0^a f(a-x)\mathrm{d}x = \left[-F(a-x)\right]_0^a = -F(0) + F(a)$ and hence result.	**SOP2** (Possible replacement for SOP3)	Any correct alternative method accepted, for example a correct transformation argument using reflection and translation.
3(b) (ii)	Using the result in 3(b)(i) with $a = \frac{\pi}{2}$ $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\sin^n\left(\frac{\pi}{2} - x\right)}{\sin^n\left(\frac{\pi}{2} - x\right) + \cos^n\left(\frac{\pi}{2} - x\right)}\mathrm{d}x$ $= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\mathrm{d}x$ since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and so $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x$ $= \frac{1}{2}\left\{\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\mathrm{d}x\right\}$ $= \frac{1}{2}\left\{\int_0^{\frac{\pi}{2}} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\mathrm{d}x\right\} = \frac{1}{2}\left\{\frac{\pi}{2}\right\} = \frac{\pi}{4}$	**SOP3**	No alternative.

No.	Evidence	Code	Judgement
4(a) (i)	$\lvert v\rvert=\sqrt{2}$ Using de Moivre's Theorem: $\lvert vz\rvert=\lvert v\rvert\lvert z\rvert=\sqrt{2}\lvert z\rvert$ So $\lvert z-v\rvert=\lvert vz\rvert$ becomes $\lvert z-v\rvert=\sqrt{2}\lvert z\rvert$ Squaring $\lvert z-v\rvert^2=2\lvert z\rvert^2$ Letting $z=x+\mathrm{i}y$ we get $(x-1)^2+(y-1)^2=2(x^2+y^2)$ and $x^2+2x+y^2+2y=2$ $(x+1)^2+(y+1)^2=4$ a circle centre (–1,–1) and radius 2. **OR** use $vz=(1+\mathrm{i})(x+\mathrm{i}y)=(x-y)+\mathrm{i}(x+y)$ So $\lvert z-v\rvert=\lvert vz\rvert$ becomes $(x-1)^2+(y-1)^2=(x-y)^2+(x+y)^2$ $-2x-2y+2=x^2+y^2$ and as above $x^2+2x+y^2+2y=2$ $(x+1)^2+(y+1)^2=4$ a circle centre (–1,–1) and radius 2.	**S2**	Either form of the circle equation will be accepted. No alternative.
4(a) (ii)	The line $\lvert z-v\rvert=\lvert z+v\rvert$ means that z lies on the perpendicular bisector of the line joining (1,1) and (–1,–1) in the complex plane, ie $y=-x$ **OR** $\lvert z-v\rvert=\lvert z+v\rvert$ so $(x-1)^2+(y-1)^2=(x+1)^2+(y+1)^2$ and $-2x-2y=2x+2y,\quad y=-x$. This meets the circle $(x+1)^2+(y+1)^2=4$ where $(x+1)^2+(-x+1)^2=4,\quad 2x^2=2$ $x^2=1,\quad x=\pm1$ and $y=\mp1$ ie at the points (1,–1) and (–1,1).	**S2**	Any correct alternative method accepted. Accept $z=\pm1\mp\mathrm{i}$

No.	Evidence	Code	Judgement
4(b)	$z^5 - 1 = 0$, $(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$ roots of $z^5 - 1 = 0$, are $z = \text{cis}\left(\frac{2k\pi}{5}\right),\ k = 0,\ 1,\ 2,\ 3,\ 4$ and $k = 0$ gives $z = 1$, so $z^4 + z^3 + z^2 + z + 1 = \left(z - \text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)$ $\left(z - \text{cis}\left(\frac{6\pi}{5}\right)\right)\left(z - \text{cis}\left(\frac{8\pi}{5}\right)\right)$ $= \left(z - \text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)$ $\left(z - \text{cis}\left(-\frac{4\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{2\pi}{5}\right)\right)$ $\left(z^2 - \left(\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right)\right)z + \text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)\right)$ $=$ $\left(z^2 - \left(\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right)\right)z + \text{cis}\left(\frac{4\pi}{5}\right)\text{cis}\left(-\frac{4\pi}{5}\right)\right)$ but $\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) = 2\cos\left(\frac{2\pi}{5}\right)$ and $\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) = 2\cos\left(\frac{4\pi}{5}\right)$ also $\text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right) = \text{cis}(0) = 1$ and $\text{cis}\left(\frac{4\pi}{5}\right)\text{cis}\left(-\frac{4\pi}{5}\right) = 1$	**SOP1**	Any correct alternative method accepted.

	so $z^4 + z^3 + z^2 + z + 1 =$ $\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$ Comparing coefficients of z^2 $1 = 1 + 1 + 4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)$ and $\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = -\frac{1}{4}$		Factorisation required. Accept $(z^2 - 0.618z + 1)(z^2 + 1.618z + 1)$ or $\left(z^2 - \left(\frac{-1+\sqrt{5}}{2}\right)z + 1\right)\left(z^2 - \left(\frac{-1-\sqrt{5}}{2}\right)z + 1\right)$ for S1. No alternative answer.

No.	Evidence	Code	Judgement
5(a)	$x = r\cos\theta \qquad y = r\sin\theta$ $= A\mathrm{e}^{\mathrm{k}\theta}\cos\theta \qquad = A\mathrm{e}^{\mathrm{k}\theta}\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta.A\mathrm{k}\mathrm{e}^{\mathrm{k}\theta} - \sin\theta.A\mathrm{e}^{\mathrm{k}\theta}$ $= A\mathrm{e}^{\mathrm{k}\theta}(\mathrm{k}\cos\theta - \sin\theta)$ $\frac{\mathrm{d}y}{\mathrm{d}\theta} = A\mathrm{k}\mathrm{e}^{\mathrm{k}\theta}\sin\theta + A\mathrm{e}^{\mathrm{k}\theta}\cos\theta$ $= A\mathrm{e}^{\mathrm{k}\theta}(\mathrm{k}\sin\theta + \cos\theta)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{A\mathrm{e}^{\mathrm{k}\theta}(\mathrm{k}\sin\theta + \cos\theta)}{A\mathrm{e}^{\mathrm{k}\theta}(\mathrm{k}\cos\theta - \sin\theta)}$ $= \frac{\mathrm{k}\sin\theta + \cos\theta}{\mathrm{k}\cos\theta - \sin\theta}$ $= \frac{\mathrm{k}\tan\theta + 1}{\mathrm{k} - \tan\theta}$	S1	Either form of $\frac{\mathrm{d}y}{\mathrm{d}x}$, or equivalent, will be accepted.

No.	Evidence	Code	Judgement
5(b)	$\frac{dy}{dx} = \tan(\alpha + \theta)$ (gradient of the tangent using exterior angle of a triangle) (see diagram) so $\frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{k\tan\theta + 1}{k - \tan\theta}$ $\frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} = \frac{\tan\theta + \frac{1}{k}}{1 - \frac{1}{k}\tan\theta}$ and $\tan\alpha = \frac{1}{k}$ $\alpha = \tan^{-1}\left(\frac{1}{k}\right)$ **OR** use $\left(\tan\alpha + \tan\theta\right)\left(k - \tan\theta\right) = \left(1 - \tan\alpha\tan\theta\right)\left(k\tan\theta + 1\right)$ $\left(k\tan\alpha - 1\right)\left(\tan^2\theta + 1\right) = 0$ $\tan\alpha = \frac{1}{k}$ since $\tan^2\theta + 1 \neq 0$ $\alpha = \tan^{-1}\left(\frac{1}{k}\right)$ **OR** $\frac{dy}{dx} = \tan(\alpha + \theta)$ and from (a) $\frac{dy}{dx} = \frac{k\sin\theta + \cos\theta}{k\cos\theta - \sin\theta}$ hence $\tan(\alpha + \theta) = \frac{k\sin\theta + \cos\theta}{k\cos\theta - \sin\theta}$ and this is true for all values of θ. In particular it is true for $\theta = 0$, and then $\tan(\alpha + 0) = \frac{k\sin 0 + \cos 0}{k\cos 0 - \sin 0} = \frac{1}{k}$ so $\tan\alpha = \frac{1}{k}$ and $\alpha = \tan^{-1}\left(\frac{1}{k}\right)$.	**SOP2**	Accept $\tan\alpha = \frac{1}{k}$, or equivalent. Accept $\frac{\pi}{2} - \tan^{-1}(k)$, or equivalent.

No.	Evidence	Code	Judgement
6(a)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=0$, integrating wrt t $\quad \frac{\mathrm{d}x}{\mathrm{d}t}=C$ but when $t=0$, $v_x = V\cos\alpha$, so $\frac{\mathrm{d}x}{\mathrm{d}t}=V\cos\alpha$ $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=-\mathrm{g}$, integrating wrt t, $\frac{\mathrm{d}y}{\mathrm{d}t}=-\mathrm{g}t+K$ but when $t=0$, $v_y = V\sin\alpha$, so $K=V\sin\alpha$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=-\mathrm{g}t+V\sin\alpha$ Integrating again $\frac{\mathrm{d}x}{\mathrm{d}t}=V\cos\alpha$ so $x=Vt\cos\alpha+M$ but $x=0$ when $t=0$, so $M=0$ and $x=Vt\cos\alpha$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-\mathrm{g}t+V\sin\alpha$ so $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha+N$ but $y=0$ when $t=0$, so $N=0$ and $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha$ From $x=Vt\cos\alpha$ we get $t=\frac{x}{V\cos\alpha}$ and substituting into $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha$ gives $y=\frac{-\mathrm{g}\left(\frac{x}{V\cos\alpha}\right)^2}{2}+V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha$, so $y=x\tan\alpha-\frac{\mathrm{g}x^2}{2V^2}\sec^2\alpha$ or $y=x\tan\alpha-\frac{\mathrm{g}x^2}{2V^2}(1+\tan^2\alpha)$	**S1** (Possible replacement for S2).	Accept parametric answer: $x=Vt\cos\alpha$ $y=\frac{-\mathrm{g}t^2}{2}+Vt\sin\alpha$ this form only.

No.	Evidence	Code	Judgement
6(b) (i)	If the centre passes through the point (kh,h), then from the above $h = kh\tan\alpha - \frac{g(kh)^2}{2V^2}(1+\tan^2\alpha)$ which is a quadratic in tan α, so rearranging $g(kh)^2\tan^2\alpha - 2V^2kh\tan\alpha + 2V^2h + g(kh)^2 = 0$ and, $h \neq 0$ $ghk^2\tan^2\alpha - 2V^2k\tan\alpha + 2V^2 + ghk^2 = 0$ This has two distinct real solutions if and only if $b^2 - 4ac > 0$ $\left(2V^2k\right)^2 - 4ghk^2(2V^2 + ghk^2) > 0$ $4V^4k^2 - 4ghk^2(2V^2 + ghk^2) > 0$ $V^4 - 2ghV^2 - g^2h^2k^2 > 0 \quad (k^2 > 0)$ the LHS of which is a quadratic in V^2, and has the form $(V^2 - a)(V^2 - b)$ where *a, b* arise from $V^2 = \frac{2gh \pm \sqrt{(2gh)^2 + 4g^2h^2k^2}}{2}$ $V^2 = \frac{gh \pm gh\sqrt{1+k^2}}{1}$ $V^2 = gh\left(1 \pm \sqrt{1+k^2}\right)$ but $1 - \sqrt{1+k^2} < 0$ so $4V^4 - 8ghV^2 - 4g^2k^2h^2 > 0$ when $V^2 > gh\left(1 + \sqrt{1+k^2}\right)$ **OR** $y = \frac{-gt^2}{2} + Vt\sin\alpha$ and $x = Vt\cos\alpha$ since it passes through the point (kh,h) $h = \frac{-gt^2}{2} + Vt\sin\alpha$ and $kh = Vt\cos\alpha$ $Vt\sin\alpha = h + \frac{gt^2}{2}$ $V^2t^2\sin^2\alpha = \left(h + \frac{gt^2}{2}\right)^2$	**SOP3** **(**Possible replacement for SOP1)	

	$V^2t^2\left(1-\left(\frac{kh}{Vt}\right)^2\right)=\left(h+\frac{gt^2}{2}\right)^2$ $4\left(V^2t^2-k^2h^2\right)=4h^2+4ght^2+g^2t^4$ $g^2t^4+4\left(gh-V^2\right)t^2+4h^2\left(1+k^2\right)=0$ and to get 2 different values of t we need $b^2-4ac>0$ $16\left(gh-V^2\right)^2-16g^2h^2\left(1+k^2\right)>0$ $\left(gh-V^2\right)^2-g^2h^2\left(1+k^2\right)>0$ $gh-V^2<-gh\sqrt{1+k^2}$ or $gh-V^2>gh\sqrt{1+k^2}$ $V^2>gh\left(1+\sqrt{1+k^2}\right)$ ($V^2<gh\left(1-\sqrt{1+k^2}\right)$ not possible since $1-\sqrt{1+k^2}<0$).		

No.	Evidence	Code	Judgement
6(b) (ii)	Use $g(kh)^2\tan^2\alpha - 2V^2kh\tan\alpha + 2V^2h + g(kh)^2 = 0$ Let $\tan\alpha_1$, $\tan\alpha_2$ be the two roots, then by the sum and the product of the roots: $\tan\alpha_1 + \tan\alpha_2 = \dfrac{2V^2kh}{gk^2h^2} = \dfrac{2V^2}{gkh}$ and $\tan\alpha_1 \times \tan\alpha_2 = \dfrac{2V^2h + gk^2h^2}{gk^2h^2} = \dfrac{2V^2 + gk^2h}{gk^2h}$ and, since $\tan(\alpha_1 + \alpha_2) = \dfrac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \times \tan\alpha_2}$ $\tan(\alpha_1 + \alpha_2) = \dfrac{\frac{2V^2}{gkh}}{1 - \frac{2V^2 + gk^2h}{gk^2h}}$ $= \dfrac{2kV^2}{gk^2h - 2V^2 - gk^2h}$ $= -k$ and $\alpha_1 + \alpha_2 = \tan^{-1}(-k)$.	**SOP2** (Possible replacement for SOP1 and SOP3)	