Assessment Schedule – 2009

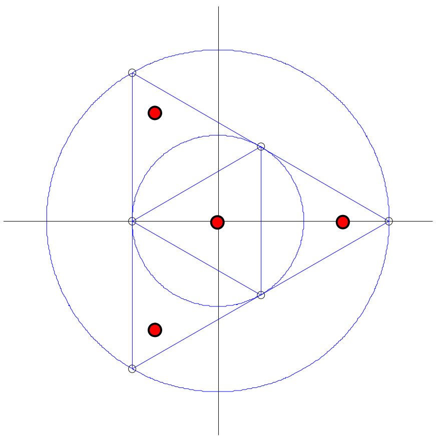
Scholarship Mathematics with Calculus (93202)

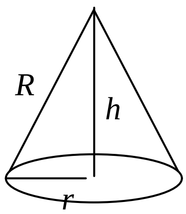
Evidence Statement

One minor error may be ignored on each part of each of the five questions (MEI)
On each question a total of 9 marks are available, with 9 dropping to 8 on any question (allowing a candidate one incomplete part for a top mark)

Question	Evidence	Code
ONE (a)	$\frac{dy}{dx} = ka\cos(kx) - kb(\sin kx)$ $\frac{d^2y}{dx^2} = -k^2 a\sin(kx) - k^2 b\cos(kx)$	**1** for any form of 2nd derivative (and 1st deriv. correct, if given)
	$= -k^2(a\sin(kx) + b\cos(kx))$ $= -k^2 y$ [Alternative method: integrate y twice, setting integration constants to zero]	**2** for required form (with y) **3** if without MEI
(b)	$s(t) = \sqrt{(1+\cos t)^2 + (-\sin t)^2}$ $= \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t}$ $= \sqrt{2 + 2\cos t}$	**1** for a **simplified** form of $s(t)$
	$= (2 + 2\cos t)^{1/2} \quad = \sqrt{2}\sqrt{1+\cos t}$ $a(t) = \frac{\mathrm{d}s}{\mathrm{d}t}$ $= \frac{-2\sin t}{2(2+2\cos t)^{1/2}}$	**+1** for any form of $a(t)$
	$= \frac{-\sin t}{\sqrt{2+2\cos t}} \quad = \frac{-\sin t}{\sqrt{2}\sqrt{1+\cos t}}$ Will be undefined when $\cos t = -1$	**+1** for any $a(t)$ discontinuity (condition for $\cos t$ sufficient)
	When $t = (2k+1)\pi$ (with k an integer), the function $a(t)$ is undefined, and so not continuous at odd multiples of π.	
(c)	$(1+\text{cis}\,2\theta)(1+\text{cis}\,4\theta) = 1 + \text{cis}\,2\theta + \text{cis}\,4\theta + \text{cis}\,6\theta$ $= 1 + \cos 2\theta + \mathrm{i}\sin 2\theta + \cos 4\theta + \mathrm{i}\sin 4\theta + \cos 6\theta + \mathrm{i}\sin 6\theta$ $= (1 + \cos 2\theta + \cos 4\theta + \cos 6\theta) + i(\sin 2\theta + \sin 4\theta + \sin 6\theta)$	**1** for separating into real and imaginary (any forms)
	$u = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$ $v = \sin 2\theta + \sin 4\theta + \sin 6\theta$ $\frac{u}{v} = \frac{1 + \cos 2\theta + \cos 4\theta + \cos 6\theta}{\sin 2\theta + \sin 4\theta + \sin 6\theta}$ $= \frac{1 + 2\cos 3\theta\cos\theta + 2\cos^2 3\theta - 1}{2\sin 3\theta\cos\theta + 2\sin 3\theta\cos 3\theta}$	**2** for double angle for 3θ
	$= \frac{\cos 3\theta(\cos\theta + \cos 3\theta)}{\sin 3\theta(\cos\theta + \cos 3\theta)}$ $= \frac{\cos 3\theta}{\sin 3\theta}$ $= \cot 3\theta$	**3** (or otherwise) arrive at answer

TWO (a)	$4\cosh x^3 - 3\cosh x = \frac{1}{2}(e^x + e^{-x})^3 - \frac{3}{2}(e^x + e^{-x})$ $= \frac{1}{2}\left(e^{3x} + 3e^{2x}e^{-x} + 3e^x e^{-2x} + e^{-3x}\right) - \frac{3}{2}e^x - \frac{3}{2}e^{-x}$ $= \frac{1}{2}e^{3x} + \frac{3}{2}e^x + \frac{3}{2}e^{-x} + \frac{1}{2}e^{-3x} - \frac{3}{2}e^x - \frac{3}{2}e^{-x}$ $= \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x} = \frac{e^{3x} + e^{-3x}}{2}$ $= \cosh(3x)$	**1** expanding cubic term **2** arrive at answer **3** without MEI
(b)	$\int_0^a \pi x^{2p}\, dx = \int_0^{a^p} \pi y^{2/p}\, d y$ $\left[\frac{\pi x^{2p+1}}{2p+1}\right]_0^a = \left[\frac{\pi y^{2/p+1}}{2/p+1}\right]_0^{a^p}$ $\frac{a^{2p+1}}{2p+1} = \frac{a^{2+p}}{2/p+1}$ $\frac{a^{2p+1}}{a^{2+p}} = \frac{2p+1}{2/p+1}$ $a^{p-1} = \frac{p(2p+1)}{2+p}$ $a = \sqrt[p-1]{\frac{p(2p+1)}{2+p}} = \left(\frac{p(2p+1)}{2+p}\right)^{1/(p-1)}$	**+1** both integrals with limits **+1** evaluate both **+1** any form for *a* (inc. log form)
(c)	The exact form is $\int_r^t (ax^2 + bx + c)\,dx = \left[\frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx\right]_r^t$ $= \frac{1}{3}a(t^3 - r^3) + \frac{1}{2}b(t^2 - r^2) + c(t - r)$ $= \frac{1}{3}at^3 + \frac{1}{2}bt^2 + ct - \frac{1}{3}ar^3 - \frac{1}{2}br^2 - cr$ The approximation is $\frac{s-r}{6}\left[y_0 + 4y_1 + y_2\right] = \frac{t-r}{6}\left[ar^2 + br + c + 4\left(a(\frac{t+r}{2})^2 + b\frac{t+r}{2} + c\right) + at^2 + bt + c\right]$ $= \frac{t-r}{6}\left[ar^2 + br + c + a(t+r)^2 + 2b(t+r) + 4c + at^2 + bt + c\right]$ $= \frac{t-r}{6}\left[2at^2 + 2ar^2 + 2atr + 3bt + 3br + 6c\right]$ $= \frac{1}{3}at^3 - \cancel{\frac{1}{3}at^2r} + \cancel{\frac{1}{3}ar^2t} - \frac{1}{3}ar^3 + \cancel{\frac{1}{3}at^2r} - \cancel{\frac{1}{3}atr^2}$ $+ \frac{1}{2}bt^2 - \cancel{\frac{1}{2}btr} + \cancel{\frac{1}{2}btr} - \frac{1}{2}br^2 + ct - cr$ $= \frac{1}{3}at^3 + \frac{1}{2}bt^2 + ct - \frac{1}{3}ar^3 - \frac{1}{2}br^2 - cr$ so they are equal, as required.	**+1** any form of integral **+1** any form of approximation **3** demonstrate equality

THREE (a)	$\frac{1}{a+bi}=\frac{1}{a}+\frac{1}{bi}$ $\frac{a-bi}{a^2+b^2}=\frac{1}{a}-\frac{1}{b}i$ Equating real and imaginary terms: $\frac{1}{a+bi}=\frac{1-bi}{a^2+b^2}$ OR $\frac{1}{a}+\frac{1}{bi}=\frac{a+bi}{abi}$ $\frac{a}{a^2+b^2}=\frac{1}{a}$ $\frac{-b}{a^2+b^2}=\frac{-1}{b}$ So $a^2=a^2+b^2$, so $b^2=0$ $-b^2=-a^2-b^2$, so $a^2=0$ So $a+bi=0+0i=0$, so there is no non-zero solution.	**1** rearrange either component **2** rearrange for any **comparable** forms **3** (or otherwise) demonstrate no valid solution
b(i)	The roots are $\left\{k, k\text{cis}\frac{2\pi}{3}, k\text{cis}\frac{4\pi}{3}, -\frac{k}{2}, \frac{k}{2}\text{cis}\frac{\pi}{3}, \frac{k}{2}\text{cis}\frac{5\pi}{3}\right\}$ $\left\{k, -\frac{1}{2}k \pm \frac{\sqrt{3}}{2}ki, -\frac{1}{2}k, \frac{1}{4}k \pm \frac{\sqrt{3}}{4}ki\right\}$ $p(x)=(x-k)(x+\frac{k}{2})q(x)$ $=A(x^3-k^3)\left(x^3+\left(\frac{k}{2}\right)^3\right)$ $=A(x-k)(x+\frac{k}{2})(x^2+kx+k^2)(x^2-\frac{k}{2}x+\frac{k^2}{4})$	**1** for **exact** roots (any form) MEI if 1 missing **2** linear terms only **3 (**or otherwise) $p(x)$ any form
b(ii)	$p'(x)=3Ax^2(x^3+\frac{k^3}{8})+3Ax^2(x^3-k^3)$ $=3Ax^2(2x^3-\frac{7}{8}k^3)=6Ax^2(x^3-\frac{7}{16}k^3)$ $=6Ax^5-\frac{21}{8}Ak^3x^2$ $p'(x)=Ax^2(6x^3-\frac{21}{8}k^3)$ $x=0$ or $6x^3=\frac{21}{8}k^3$ $x^3=\frac{7}{16}k^3$ The roots of $p'(x)$ are $\left\{0,0,\sqrt[3]{\frac{7}{16}}k, \sqrt[3]{\frac{7}{16}}k\text{cis}\frac{2\pi}{3}, \sqrt[3]{\frac{7}{16}}k\text{cis}\frac{4\pi}{3}\right\}$(and note that $\sqrt[3]{\frac{7}{16}}=\frac{\sqrt[3]{28}}{4}$) 	**1** derivative (any form) but: Consistency mark *NOT* available if $p(x)$ is too simple **2** for **exact** roots (any form) **3** for radial positions of roots

FOUR (a)	$\frac{dy}{dx} = \frac{-3}{2}x^{1/2}$ $L = \int_0^{k^{2/3}} \sqrt{1+\frac{9}{4}x}\,dx$ $= \left[\frac{2}{3}\,\frac{4}{9}\left(1+\frac{9}{4}x\right)^{3/2}\right]_0^{k^{2/3}}$	**+1** correct integral for line length: limits **not required**
	$= \frac{8}{27}\left(1+\frac{9}{4}k^{2/3}\right)^{3/2} - \frac{8}{27}$ $\frac{56}{27} = \frac{1}{27}\left(4+9k^{2/3}\right)^{3/2} - \frac{8}{27}$	**+1** set equal to 56/27, with limits 0 and $k^{2/3}$ (or b where used to find k)
	$64 = \left(4+9k^{2/3}\right)^{3/2}$ $16 = 4+9k^{2/3}$ $12 = 9k^{2/3}$ $k^{2/3} = \frac{4}{3}$ $k = \frac{8}{3\sqrt{3}} = \frac{8}{9}\sqrt{3}$	**+1** correct value in surd form
(b)	Arc length of sector of circle = $R\theta$ Volume of the cone = $V = \frac{1}{3}\pi r^2 h$ Circumference of circle = arc length $2\pi r = R\theta \Rightarrow r = \frac{R\theta}{2\pi}$  	**1** relationshipof θ to circumferenceor radius
	$r^2 + h^2 = R^2 \Rightarrow h = \sqrt{R^2 - r^2} = \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2}$ Substituting $V = \frac{1}{3}\pi\left(\frac{R\theta}{2\pi}\right)^2 \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2}$	**2** volume in terms of one variable
	$= \frac{R^3}{24\pi^2}\left(\theta^2\right)\sqrt{4\pi^2 - \theta^2}$ $\frac{dV}{d\theta} = \frac{R^3}{24\pi^2}\left[2\theta\sqrt{4\pi^2-\theta^2} + \theta^2\,\frac{1}{2}\left(4\pi^2-\theta^2\right)^{\frac{-1}{2}}\left(-2\theta\right)\right] = 0$ $2\sqrt{4\pi^2-\theta^2} = \theta^2\left(4\pi^2-\theta^2\right)^{\frac{-1}{2}}$ $4\pi^2 - \theta^2 = \frac{\theta^2}{2}$ $\theta = \sqrt{\frac{8}{3}}\pi$	**3** angle for max. volume: exact not required

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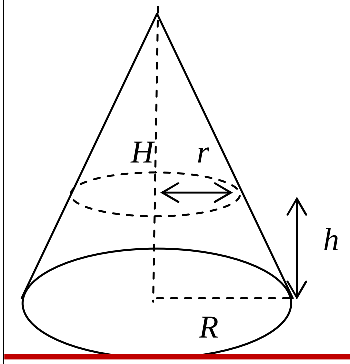

$$\frac{H-h}{r}=\frac{H}{R}$$

$$H-h=\frac{r}{R}H$$

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$$V=\frac{1}{3}\pi R^2H-\frac{1}{3}\pi r^2\left(H-h\right)$$

$$=\frac{1}{3}\pi R^2H-\frac{1}{3}\pi r^2\left(\frac{rH}{R}\right)$$

$$=\frac{1}{3}\pi R^2H-\frac{H}{3R}\pi r^3$$

$$\frac{dV}{dr}=-\frac{\pi Hr^2}{R}$$

$$\frac{dh}{dt}=\frac{dh}{dr}\times\frac{dr}{dV}\times\frac{dV}{dt}$$

$$=\frac{-H}{R}\times\frac{-R}{\pi Hr^2}\times k\pi r^2=k$$

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1 relationship of heights/radii in cone and frustum (relative to candidate’s variables)

2 use Chain rule to find $dh/dt = k$

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$$h=\int dh=\int k\,dt=kt+C$$

Initially $h=\frac{1}{2}H$ at $t=0$ giving $C=\frac{1}{2}H$

Then $h=0$ at $t=10$ giving

$$10k+\tfrac{1}{2}H=0$$

$$k=-\tfrac{1}{20}H$$

$$h=-\tfrac{1}{20}Ht+\tfrac{1}{2}H$$

Half volume occurs at $h=H\left(1-\frac{1}{\sqrt[3]{2}}\right)$

$$H\left(1-\frac{1}{\sqrt[3]{2}}\right)=-\tfrac{1}{20}Ht+\tfrac{1}{2}H$$

$$1-\frac{1}{\sqrt[3]{2}}=-\tfrac{1}{20}t+\tfrac{1}{2}$$

$$\tfrac{1}{20}t=\frac{1}{\sqrt[3]{2}}-\frac{1}{2}$$

$$t=\frac{20}{\sqrt[3]{2}}-10$$

[Approximately 5.874 days]

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3 correct value: exact not required

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(a)</td><td>Area of square $A_S = (\sqrt{2}b)^2 = 2b^2$
Area of ellipse $A_E = \pi ab$

$4b^2 = \pi ab$</td><td>1 relate areas in terms of a and b</td></tr>
<tr><td></td><td>$a = \frac{4}{\pi} b$
$b = \frac{\pi}{4} a$
$c^2 = a^2 - b^2 = a^2 - \frac{\pi^2}{16} a^2$
$e = \frac{c}{a} = \sqrt{1 - \frac{\pi^2}{16}}$</td><td>2 eccentricity still has a or b

3 correct value (exact value not required)</td></tr>
<tr><td>(b)</td><td>$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$
$\frac{xx_0}{a^2} = 1 + \frac{yy_0}{b^2}$
$x = \frac{a^2}{x_0}\left(1 + \frac{y_0}{b^2} y\right)$
$x = \frac{a^2b^2 + a^2y_0y}{b^2x_0}$

Alternative (asymptotes):
1 for a graphical understanding of approaching asymptote
2 for more carefully explained understanding
3 for a full explanation, including limit of gradient and diagram</td><td>1 make x or y the subject of either equation</td></tr>
<tr><td></td><td>Substituting into the equation for the hyperbola
$b^2x^2 - a^2y^2 = a^2b^2$:
$b^2\left(\frac{a^2b^2 + a^2y_0y}{b^2x_0}\right)^2 - a^2y^2 = a^2b^2$
$a^2\left(b^4 + 2b^2y_0y + y_0^2y^2\right) - b^2x_0^2y^2 = b^4x_0^2$
$y^2\left(a^2y_0^2 - b^2x_0^2\right) + 2a^2b^2y_0y + b^4\left(a^2 - x_0^2\right) = 0$</td><td>2 for quadratic in y or x</td></tr>
<tr><td></td><td>$-a^2b^2y^2 + 2a^2b^2y_0y + b^4\left(a^2 - x_0^2\right) = 0$
$-a^2y^2 + 2a^2y_0y + b^2\left(a^2 - x_0^2\right) = 0$
A quadratic in y
$y^2 - 2y_0y - b^2\left(1 - \frac{x_0{}^2}{a^2}\right) = 0$
$y^2 - 2y_0y - b^2\left(-\frac{y_0{}^2}{b^2}\right) = 0$
$y^2 - 2y_0y + y_0{}^2 = (y - y_0)^2 = 0$
The quadratic has one (repeated) root; there are no other points of intersection.</td><td>3 show (x_0, y_0) is the only solution</td></tr>
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(c)	*First method* Starting from the general tangent to the ellipse $y = mx \pm \sqrt{a^2m^2 + b^2}$ $y - mx = \pm\sqrt{a^2m^2 + b^2}$ $y^2 - 2mx + m^2x^2 = a^2m^2 + b^2$ This equation can be rearranged to form a quadratic equation in m, the gradient of the tangents. $m^2\left(x^2 - a^2\right) - 2xym + y^2 - b^2 = 0$ Since the tangents are perpendicular the product of the roots of the equation is –1 and hence $\frac{y^2 - b^2}{x^2 - a^2} = -1$ $y^2 - b^2 = a^2 - x^2$ $x^2 + y^2 = a^2 + b^2$ *Second Method* Starting from the general tangent to the ellipse. Gradient of the first tangent is m_1 $y = m_1x \pm \sqrt{a^2m_1^{\ 2} + b^2}$ $y - m_1x = \pm\sqrt{a^2m_1^{\ 2} + b^2}$ $y^2 - 2m_1x + m_1^{\ 2}x^2 = a^2m_1^{\ 2} + b^2$ (1)	**+1** any quadratic expression in *m*
	Similarly the equation of the second tangent with gradient m_2 is $y^2 - 2m_2x + m_2^{\ 2}x^2 = a^2m_2^{\ 2} + b^2$ As the tangents are perpendicular $m_1 \times m_2 = -1 \Rightarrow m_2 = \frac{-1}{m_1}$	**+1** importance of $m_1m_2 = -1$, stated or used (as above)
	Substituting for m_2 gives $y^2 - 2\left(\frac{-1}{m_1}\right)x + \left(\frac{-1}{m_1}\right)^2 x^2 = a^2\left(\frac{-1}{m_1}\right)^2 + b^2$ $\Rightarrow m_1^{\ 2}y^2 + 2m_1x + x^2 = a^2 + m_1^{\ 2}b^2$ (2) Adding equations 1 and 2 $y^2\left(1 + m_1^{\ 2}\right) + x^2\left(1 + m_1^{\ 2}\right) = a^2\left(1 + m_1^{\ 2}\right) + b^2\left(1 + m_1^{\ 2}\right)$ $y^2 + x^2 = a^2 + b^2$	**3** show that locus of Q is a circle: (equation not required)
	The circle is centred at the origin, with radius $\sqrt{a^2 + b^2}$	

The question responses were awarded up to a total of 8 marks where candidates showed they were working at outstanding scholarship level.