Assessment Schedule – 2010

Scholarship Mathematics with Calculus (93202)

Evidence Statement

ONE (a)	Using de Moivre's Theorem and trigonometric identities: $$\begin{aligned} z^{19}+z^{14}+z^{8}+z^{3} &= \left(\operatorname{cis}\tfrac{7\pi}{11}\right)^{19}+\left(\operatorname{cis}\tfrac{7\pi}{11}\right)^{14}+\left(\operatorname{cis}\tfrac{7\pi}{11}\right)^{8}+\left(\operatorname{cis}\tfrac{7\pi}{11}\right)^{3} \\ &= \operatorname{cis}\tfrac{133\pi}{11}+\operatorname{cis}\tfrac{98\pi}{11}+\operatorname{cis}\tfrac{56\pi}{11}+\operatorname{cis}\tfrac{21\pi}{11} \\ &= \operatorname{cis}\left(12\pi+\tfrac{\pi}{11}\right)+\operatorname{cis}\left(8\pi+\tfrac{10\pi}{11}\right)+\operatorname{cis}\left(6\pi-\tfrac{10\pi}{11}\right)+\operatorname{cis}\left(2\pi-\tfrac{\pi}{11}\right) \\ &= \operatorname{cis}\left(\tfrac{\pi}{11}\right)+\operatorname{cis}\left(\tfrac{10\pi}{11}\right)+\operatorname{cis}\left(-\tfrac{10\pi}{11}\right)+\operatorname{cis}\left(-\tfrac{\pi}{11}\right) \\ &= \operatorname{cis}\left(\tfrac{\pi}{11}\right)+\operatorname{cis}\left(\pi-\tfrac{\pi}{11}\right)+\operatorname{cis}\left(\tfrac{\pi}{11}-\pi\right)+\operatorname{cis}\left(-\tfrac{\pi}{11}\right) \\ &= \operatorname{cis}\left(\tfrac{\pi}{11}\right)-\operatorname{cis}\left(-\tfrac{\pi}{11}\right)-\operatorname{cis}\left(\tfrac{\pi}{11}\right)+\operatorname{cis}\left(-\tfrac{\pi}{11}\right) \\ &= 0 \end{aligned}$$	**1** (maximum): calculator evaluates to zero. **1** use de Moivre's Theorem. **2** show, any way. **3** show using trig identities.
(b)	First, note that $y=\sin^2 x = 1-\cos^2 x$, so $\cos^2 x = 1-y$. Then $$\begin{aligned} &12y^3-13y^2-14y+13=1-y \\ &12y^3-13y^2-13y+12=0 \\ &(y+1)(4y-3)(3y-4)=0 \end{aligned}$$ So $y=-1,\frac{4}{3},\frac{3}{4}$. Now since $\sin x=\sqrt{y}$, only $\sin x=\frac{\sqrt{3}}{2}$ is possible, yielding $x=\frac{\pi}{3}+2n\pi,\frac{2\pi}{3}+2n\pi$ for integer n, or $x=n\pi+(-1)^n\dfrac{\pi}{3}$	**1** factorise or find roots. **2** only $\sin x=\frac{\sqrt{3}}{2}$ **3** all solutions, any form.
(c)	The equation further factorises and simplifies, with some effort, to $$\frac{(z-1)(z+1)(z^2+z+1)(z^2+1)(z^2-z+1)(z^4-z^2+1)}{(z-1)^2(z+1)(z^2+z+1)(z^2+1)}=0$$ $$\frac{(z^2-z+1)(z^4-z^2+1)}{z-1}=0$$ and the following answer can be reached from here, with a little more brute force (polar form is not required in the answer). Alternatively, the roots of the numerator are, in polar form, $\operatorname{cis}\frac{k\pi}{6}$ for integers $0\le k<12$. We then rule out the roots of the denominator, which are $\operatorname{cis}\frac{4m\pi}{6}$ and $\operatorname{cis}\frac{3n\pi}{6}$ for integers m, n. This leaves the six roots $\operatorname{cis}\frac{\pi}{6},\operatorname{cis}\frac{2\pi}{6},\operatorname{cis}\frac{5\pi}{6},\operatorname{cis}\frac{7\pi}{6},\operatorname{cis}\frac{10\pi}{6},\operatorname{cis}\frac{11\pi}{6}$ or $\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$ and $\pm\frac{\sqrt{3}}{2}\pm\frac{1}{2}\mathrm{i}$ (or $\pm\sqrt{\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}}$).	**1** factorise and eliminate at least one denominator term. **2** fully factorised form. **1** find all twelve roots of numerator. **2** work to rule out denom roots. **3** these <u>six</u> roots, <u>no</u> <u>others</u> (any form).

TWO (a)(i)	The points on the x-axis touching the inscribed circle are on the hyperbola $x^2 - y^2 = 2\alpha$ with $y = 0$ and so $x^2 = 2\alpha$. The circle has radius $\sqrt{2\alpha}$ and so has area $A_{\text{inscribed}} = 2\alpha\pi \approx 15.1690$. The points where the circumscribed circle touches the hyperbolae include $(\alpha, 1)$. The radius r of the circumscribed circle is then given by $\begin{aligned} r^2 &= x^2 + y^2 \\ &= \alpha^2 + 1 \\ &= (1 + 2\sqrt{2} + 2) + 1 \\ &= 4 + 2\sqrt{2} \\ A_{\text{circumscribed}} &= \left(4 + 2\sqrt{2}\right)\pi \\ &\approx 21.4521 \end{aligned}$	**+1** area, any form. **+1** ANY intersection: $(\pm 1, \pm\alpha)$ or $(\pm\alpha, \pm 1)$ **+1** area, any form
(ii)	We find (as above) that the x-coordinates of intersections are $\pm 1, \pm\alpha$. Hence we find that the area is $A = 8\int_1^\alpha \frac{\alpha}{x}\,\mathrm{d}x = 8\alpha\left[\ln x\right]_1^\alpha = 8\alpha\left(\ln\alpha - \ln 1\right) = 8\alpha\ln\alpha = 8\left(1+\sqrt{2}\right)\ln\left(1+\sqrt{2}\right)$ $= 16\alpha\ln\sqrt{\alpha} = 17.0226$	**+1** ANY intersection (unless awarded above). **1** correct integral with limits. **2** area, any form.
(b)	The first and second derivatives are $\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{1}{3}x^{-2/3}\mathrm{e}^{-x^2} + x^{1/3}(-2x)\mathrm{e}^{-x^2} \\ &= \tfrac{1}{3}\mathrm{e}^{-x^2}\left(x^{-2/3} - 6x^{4/3}\right) \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \tfrac{1}{3}\mathrm{e}^{-x^2}\left(-\tfrac{2}{3}x^{-5/3} - 8x^{1/3}\right) + \tfrac{1}{3}(-2x)\mathrm{e}^{-x^2}\left(x^{-2/3} - 6x^{4/3}\right) \\ &= \tfrac{1}{9}\mathrm{e}^{-x^2}\left(-2x^{-5/3} - 24x^{1/3} - 6x^{1/3} + 36x^{7/3}\right) \\ &= \tfrac{1}{9}\mathrm{e}^{-x^2}x^{-5/3}\left(36x^4 - 30x^2 - 2\right) \end{aligned}$ The second derivative is zero at: $x^2 = \frac{30 \pm \sqrt{900 + 288}}{72} = \frac{30 \pm \sqrt{1188}}{72} = \frac{5 \pm \sqrt{33}}{12}$ We take only the positive value, and then $x = \pm\sqrt{\frac{5+\sqrt{33}}{12}} \approx \pm 0.9462$. Finally note that the function is defined at $x = 0$ although the first and second derivatives are not. At this point, the function's concavity also changes. We can check that the second derivative changes sign at these three points by evaluating $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at four points surrounding them: $y''(1) = 0.1635 \quad y''(0.5) = -1.9918 \quad y''(-0.5) = 1.9918 \quad y''(-1) = -0.1635$	**1** correct form up to constant $= \left(4x^4 - \tfrac{10}{3}x^2 - \tfrac{2}{9}\right)\ldots$ **2** correct <u>REAL</u> roots of second derivative. **3** point of inflection at $x = 0$ AND testing all points.

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(a)(i)</td><td>

(a)(i) Using the given equation and the definition of $r(\theta)$ we find

$$\begin{aligned}
A &= \tfrac{1}{2}\int_0^{2\pi} r^2 \,\mathrm{d}\theta \\
&= \tfrac{1}{2}\int_0^{2\pi} (a + b\sin(n\theta))^2 \,\mathrm{d}\theta \\
&= \tfrac{1}{2}\int_0^{2\pi} \left(a^2 + 2ab\sin(n\theta) + b^2\sin^2(n\theta)\right)\mathrm{d}\theta \\
&= \tfrac{1}{2}\int_0^{2\pi} \left(a^2 + 2ab\sin(n\theta) + \tfrac{1}{2}b^2(1 - \cos(2n\theta))\right)\mathrm{d}\theta \\
&= \tfrac{1}{2}\left[a^2\theta - \tfrac{2ab}{n}\cos(n\theta) + \tfrac{1}{2}b^2\theta - \tfrac{b^2}{4n}\sin(2n\theta)\right]_0^{2\pi} \\
&= \tfrac{1}{2}\left[\left(2\pi a^2 - \tfrac{2ab}{n}\cos(2n\pi) + \pi b^2 - \tfrac{b^2}{4n}\sin(4n\pi)\right) - \left(-\tfrac{2ab}{n}\cos(0) - \tfrac{b^2}{4}\sin(0)\right)\right] \\
&= \tfrac{1}{2}\left[2\pi a^2 - \tfrac{2ab}{n} + \pi b^2 - \left(-\tfrac{2ab}{n}\right)\right] \\
&= \pi\left(a^2 + \tfrac{1}{2}b^2\right)
\end{aligned}$$

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1 trig identity or otherwise integrate $\sin^2(n\theta)$.

2 arrive at answer.

3 without any minor error.

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<tr><td>(ii)</td><td>

Working with the area from a(i) and the given relationships:

$$\begin{aligned}
A &= \left(\frac{H-h}{H}\right)^2 \pi a_0^2 + \frac{1}{2}\frac{h^2}{H^2}\left(\frac{H-h}{H}\right)^2 \pi a_0^2 \\
&= \frac{\pi a_0^2}{H^4}\left(H^4 - 2H^3h + \tfrac{3}{2}H^2h^2 - Hh^3 + \tfrac{1}{2}h^4\right)
\end{aligned}$$

$$\begin{aligned}
V &= \int_0^H A\,\mathrm{d}h \\
&= \frac{\pi a_0^2}{H^4}\left[H^4h - H^3h^2 + \tfrac{1}{2}H^2h^3 - \tfrac{1}{4}Hh^4 + \tfrac{1}{10}h^5\right]_0^H \\
&= \frac{\pi a_0^2}{H^4}\left(H^5 - H^5 + \tfrac{1}{2}H^5 - \tfrac{1}{4}H^5 + \tfrac{1}{10}H^5\right) \\
&= \pi a_0^2 H\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{10}\right) \\
&= \tfrac{7}{20}\pi a_0^2 H
\end{aligned}$$

The volume of a cone with the same base and height is $V_c = \frac{1}{3}\pi a_0^2 H$.

The ratio of these volumes is $V / V_c = (\frac{7}{20}\pi a_0^2 H) \div (\frac{1}{3}\pi a_0^2 H) = \frac{21}{20} = 1.05$, as required.

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+1 A in terms of h.

+1 correct definite integral with limits.

3 show 5% more.

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(b)	We need $\Delta \geq 0$. $\Delta = p^2(\sqrt{3}\sin\alpha + \cos\alpha)^2 - 4p^2\sin^2\alpha \geq 0$ $3\sin^2\alpha + 2\sqrt{3}\sin\alpha\cos\alpha + \cos^2\alpha - 4\sin^2\alpha \geq 0$ $2\sqrt{3}\sin\alpha\cos\alpha + 2\cos^2\alpha - 1 \geq 0$ $\sqrt{3}\sin(2\alpha) + \cos(2\alpha) \geq 0$ We find equality at $\tan(2\alpha) = \frac{-1}{\sqrt{3}}$, so $\alpha = \frac{5\pi}{12}, \frac{11\pi}{12}$. Testing intervals, $\Delta \geq 0$ when $0 \leq \alpha \leq \frac{5\pi}{12}$ and $\frac{11\pi}{12} \leq \alpha \leq \pi$.	**+1** "show that" shown. **+1** $\alpha = \frac{5\pi}{12}, \frac{11\pi}{12}$ **+1** both intervals, **exact required**.
FOUR (a)	The parabolas $y = bx^2 - k$ and $y = (x-a)^2$ meet with the same slope: $2(x-a) = 2bx$, so $x = \dfrac{a}{1-b}$. Symmetry gives the other point of intersection of $y = bx^2 - k$ with $y = (x+a)^2$ at $x = \dfrac{a}{b-1}$. $y = \left(\dfrac{a}{1-b} - a\right)^2 = \left(\dfrac{ab}{1-b}\right)^2 = \dfrac{a^2b^2}{(1-b)^2}$ and $y = b\left(\dfrac{a}{1-b}\right)^2 - k = \dfrac{a^2b}{(1-b)^2} - k$ $\dfrac{a^2b}{(1-b)^2} - k = \dfrac{a^2b^2}{(1-b)^2}$ $k = \dfrac{a^2b}{(1-b)^2} - \dfrac{a^2b^2}{(1-b)^2}$ $= \dfrac{a^2b}{(1-b)^2}(1-b)$ $= \dfrac{a^2b}{1-b}$ Alternatively, look for a repeated solution of $(x-a)^2 = bx^2 - k$; so discriminant is zero. $(1-b)x^2 - 2ax + a^2 - k = 0$ $(-2a)^2 + 4(a^2 - k)(1-b) = 0$ $-4k(1-b) = 4a^2(1-b) - 4a$ $k = \dfrac{4a^2b}{4-4b} = \dfrac{a^2b}{1-b} = \dfrac{a^2}{1-b} - a^2$	**1** either point of intersection. **1** discriminant set to zero. **2** correct answer *any* form. **3** simplified form.

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Using completing the square twice, we find the radius of the circle (r):

$$(x+g)^2+(y+f)^2=g^2+f^2-c=r^2,$$

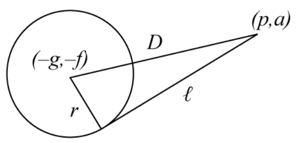
and also note that the centre of the circle is $(-g,-f)$.

The distance from the centre to the point (p,q) is
$D=\sqrt{(p-(-g))^2+(q-(-f))^2}=\sqrt{(p+g)^2+(q+f)^2}$.

Using the Pythagorean theorem on the triangle in the diagram, we find

$$\begin{aligned}\ell^2&=(p+g)^2+(q+f)^2-g^2-f^2+c\\&=p^2+2gp+g^2+q^2+2fq+f^2-g^2-f^2+c\\&=p^2+2gp+q^2+2fq+c\\\ell&=\sqrt{p^2+2gp+q^2+2fq+c}\end{aligned}$$

A diagram is required:



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1 find centre of circle.

2 or otherwise show given.

+1 diagram with right triangle.
+1 with ℓ
and two other features *labelled.*
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(c)	The gradient of the tangent line to the curve $y^2 = 4ax$ is $\dfrac{dy}{dx} = \dfrac{2a}{y} = \dfrac{2a}{2\sqrt{ax}} = \sqrt{\dfrac{a}{x}}$. At a point $\left(x_0, 2\sqrt{ax_0}\right)$ the tangent line is $y - 2\sqrt{ax_0} = \sqrt{\dfrac{a}{x_0}}(x - x_0)$ $y = \sqrt{\dfrac{a}{x_0}}x - \sqrt{\dfrac{a}{x_0}}x_0 + 2\sqrt{ax_0} = \sqrt{\dfrac{a}{x_0}}x + \sqrt{ax_0}$ The equation for the related tangent line at $x_1 = x_0 + 2k$ is then $y = \sqrt{\dfrac{a}{x_0 + 2k}}x + \sqrt{a(x_0 + 2k)}$. Equating these, (and then dividing by $\sqrt{a}$) we find the following, and then cross-multiply and square both sides to eliminate roots: $\sqrt{\dfrac{a}{x_0}}x + \sqrt{ax_0} = \sqrt{\dfrac{a}{x_0 + 2k}}x + \sqrt{a(x_0 + 2k)} = y_P$ $(x + x_0)\sqrt{x_0 + 2k} = (x + x_0 + 2k)\sqrt{x_0}$ $(x + x_0)^2(x_0 + 2k) = x_0(x + x_0 + 2k)^2$ $x_0x^2 + 2kx^2 + 2x_0^2x + 4kx_0x + x_0^3 + 2kx_0^2 = 4k^2x_0 + 4kx_0x + x_0x^2 + 2x_0^2x + x_0^3$ $2kx^2 = 4k^2x_0 + 2kx_0^2$ $x^2 = 2kx_0 + x_0^2$ $x_\mathrm{P} = \sqrt{x_0(2k + x_0)} = \sqrt{x_0x_1}$ $x_\mathrm{P}^2 = x_0(2k + x_0)$ $x_0^2 + 2kx_0 - x_\mathrm{P}^2 = 0$ $x_0 = \sqrt{k^2 + x_\mathrm{P}^2} - k$ (choosing the positive root)	**1** equation for tangent in $y =$ form in terms of x_0 or x_1
	The new curve is then $y_P = \sqrt{\dfrac{a}{x_0}}x_\mathrm{P} + \sqrt{ax_0}$, which needs to be in terms of x_P only. $y_\mathrm{P} = \sqrt{\dfrac{a}{\sqrt{k^2 + x_\mathrm{P}^2} - k}}x_P + \sqrt{a\left(\sqrt{k^2 + x_\mathrm{P}^2} - k\right)}$ $= x_P\sqrt{\dfrac{a\left(\sqrt{k^2 + x_\mathrm{P}^2} + k\right)}{x_\mathrm{P}^2}} + \sqrt{a\left(\sqrt{k^2 + x_\mathrm{P}^2} - k\right)}$ $= \sqrt{a\left(\sqrt{k^2 + x_\mathrm{P}^2} + k\right)} + \sqrt{a\left(\sqrt{k^2 + x_\mathrm{P}^2} - k\right)}$ $y_\mathrm{P}^2 = a\left(\sqrt{k^2 + x_\mathrm{P}^2} + k\right) + a\left(\sqrt{k^2 + x_\mathrm{P}^2} - k\right) + 2a\sqrt{\left(\sqrt{k^2 + x_\mathrm{P}^2} + k\right)\left(\sqrt{k^2 + x_\mathrm{P}^2} - k\right)}$ $= 2a\sqrt{k^2 + x_\mathrm{P}^2} + 2a\sqrt{k^2 + x_\mathrm{P}^2 - k^2}$ $= 2a\left(\sqrt{k^2 + x_\mathrm{P}^2} + x_\mathrm{P}\right)$	**2** expression with only x_P <u>in any form</u>.
	Now note that $4ax_\mathrm{P} < 2a\left(x_\mathrm{P} + \sqrt{x_\mathrm{P}^2 + k^2}\right) < 2a\left(x_\mathrm{P} + \sqrt{x_\mathrm{P}^2 + 2kx_\mathrm{P} + k^2}\right) = 2a(x_\mathrm{P} + x_\mathrm{P} + k) = 4ax_\mathrm{P} + 2ak$ (when $x \geq 0$).	**3** show within bounds.

FIVE (a)	A person of age T will date above the age $L(T)=\frac{1}{2}T+7$. Since the person they date also follows this rule, the upper range U can be found from $T=\frac{1}{2}U(T)+7$, so $U(T)=2T-14$. The width of the dateable range of a person of age T is then $U(T)-L(T)=\frac{3}{2}T-21$.	**1** upper range. **2** width or range
(b)	The (relative) size of the dating pool for a person of age T is then $D(T)=\int_{L(T)}^{U(T)} S(T)\,\mathrm{d}t$ $=\int_{\frac{1}{2}T+7}^{2T-14} \mathrm{e}^{-0.05t}\,\mathrm{d}t$ $=\left[-20\mathrm{e}^{-0.05t}\right]_{\frac{1}{2}T+7}^{2T-14}$ $=-20\mathrm{e}^{-0.1T+0.7}+20\mathrm{e}^{-0.025T-0.35}$ $=20\mathrm{e}^{\frac{-1}{20}(2T-14)}-20\mathrm{e}^{\frac{-1}{20}(\frac{1}{2}T+7)}$	**1** formulate. **2** integrate correctly. **3** correct (any form).
(c)	We look to find where $\frac{\mathrm{d}D}{\mathrm{d}T}=0$: $\frac{\mathrm{d}D}{\mathrm{d}T}=2\mathrm{e}^{-0.1T+0.7}-0.5\mathrm{e}^{-0.025T-0.35}$ $=2\mathrm{e}^{-0.025T-0.35}\left(\mathrm{e}^{-0.075T+1.05}-0.25\right)=0$ $\mathrm{e}^{-0.075T+1.05}=0.25$ $-0.075T+1.05=\ln 0.25$ $-0.075T=\ln 0.25-1.05$ $T=14-\frac{\ln 0.25}{0.075}=14+\frac{40}{3}\ln 4\approx 32.5$ The dating pool is largest for singles aged 32.5. This is the only age for which $\frac{\mathrm{d}D}{\mathrm{d}T}=0$.	**1** find $\frac{\mathrm{d}D}{\mathrm{d}T}$ **2** simplify or otherwise take logs. **3** correct age ANY FORM.