Assessment Schedule – 2011

Scholarship Mathematics with Calculus (93202)

Evidence Statement

QUESTION ONE SOLUTION

(a) The point (if it exists) where $\dfrac{ae^x}{2e^x-1}=1$ can be found:

$$ae^x = 2e^x - 1$$
$$(2-a)e^x = 1$$
$$e^{-x} = 2-a$$
$$x = -\ln(2-a) = \ln\frac{1}{2-a}$$

So the point exists when $0 < a < 2$.

Note that $2e^x - 1 < 0$ when $x < \ln\frac{1}{2}$ and $ae^x > 0$ always; so for any positive a, $\dfrac{ae^x}{2e^x-1} < 1$ for $x < \ln\frac{1}{2}$.

Also, if $0 < a < 2$, then $\dfrac{ae^x}{2e^x-1} < 1$ for $x > -\ln(2-a)$; positive solutions only exist for these values of a.

(b) The total surface area is $T = n_A A_A + n_B A_B = n_A\sqrt{2k^2-2k+1} + n_B\dfrac{\sqrt{3}}{2}\left(k^2-k+1\right)$, where n_A and n_B are positive integers.

$$\frac{dT}{dk} = \frac{n_A(4k-2)}{2\sqrt{2k^2-2k+1}} + n_B\frac{\sqrt{3}}{2}(2k-1)$$
$$= (2k-1)\left(\frac{n_A}{\sqrt{2k^2-2k+1}} + n_B\frac{\sqrt{3}}{2}\right)$$

The second term is always positive, so the only critical point is at $k = \frac{1}{2}$.

Using the 1DT: $\left.\dfrac{dT}{dk}\right|_{k=0} = -n_A - n_B\dfrac{\sqrt{3}}{2} < 0$ and $\left.\dfrac{dT}{dk}\right|_{k=1} = n_A + n_B\dfrac{\sqrt{3}}{2} > 0$, so we have a minimum.

(c) Using separation of variables:

$$\frac{dy}{dx} = y^{m+1}$$
$$\int y^{-m-1}dy = \int dx$$
$$\frac{y^{-m}}{-m} = x + C \qquad [\text{as } m \neq 0]$$
$$y^m = \frac{1}{m(k-x)}$$
$$y = \frac{1}{\sqrt[m]{m(k-x)}}$$

Using the chain rule, $\dfrac{d}{dx}\left(y^n\right) = \dfrac{d}{dy}\left(y^n\right)\dfrac{dy}{dx} = ny^{n-1}y^{m+1} = ny^{n+m}$, as required.

It is possible to show that the functions satisfy the property directly, but this work is not required.

QUESTION TWO SOLUTION

(a) We aim to find $\left.\dfrac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=3}$ and we see that $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 0.015 \times 25\pi = 0.375\pi$. Also $\dfrac{\mathrm{d}V}{\mathrm{d}x} = -25\pi + \pi x^2$.

$$\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt} = \frac{dV}{dt} \div \frac{dV}{dx} = \frac{0.375\pi}{(x^2-25)\pi} = \frac{0.375}{x^2-25}.$$

$\left.\dfrac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=3} = \dfrac{0.375}{9-25} = -0.0234375$ metres per hour. The water is rising at (approximately) **23.4 mm per hour**.

(b)(i) Differentiating:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(A\left(1-\sqrt{x}\right)^{3/2}\left(2+3\sqrt{x}\right)+C \right) &= A\left(1-\sqrt{x}\right)^{3/2}\frac{3}{2\sqrt{x}} - \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left(2+3\sqrt{x}\right) \\
&= \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left[2\left(1-\sqrt{x}\right)-\left(2+3\sqrt{x}\right)\right] \\
&= \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left[2-2\sqrt{x}-2-3\sqrt{x}\right] \\
&= \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left(-5\sqrt{x}\right) \\
&= -\frac{15}{4}A\sqrt{1-\sqrt{x}}
\end{aligned}$$

and so $A = -\dfrac{4}{15}$, and the Fundamental Theorem of Calculus gives the integral as required.

(b)(ii) The area under the curve $y = g(x)$ is $\displaystyle\int_0^1 g(x)\,\mathrm{d}x = \left[-\frac{4}{15}\left(1-\sqrt{x}\right)^{3/2}\left(2+3\sqrt{x}\right)\right]_0^1 = \frac{8}{15}$.

The area beneath the dotted curve is the same as the area between $y = g(x)$ and $y = 1$; that is, $1 - \dfrac{8}{15} = \dfrac{7}{15}$.

So the area between the curves is $\dfrac{1}{15}$.

QUESTION THREE SOLUTION

(a) Applying angle sum formula:

$$\cos\left(\frac{7\pi}{12}\right)=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)=\frac{1}{2}\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}=\frac{1-\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

(b) First, $\cos\theta=\dfrac{1}{\sqrt{\left(20\sqrt{6}\right)^2+1}}=\dfrac{1}{\sqrt{2401}}=\dfrac{1}{49}$.

Then $\cos\dfrac{\theta}{2}=\sqrt{\dfrac{1+\frac{1}{49}}{2}}=\sqrt{\dfrac{25}{49}}=\dfrac{5}{7}$. Now $\cos\dfrac{\theta}{4}=\sqrt{\dfrac{1+\frac{5}{7}}{2}}=\sqrt{\dfrac{6}{7}}$, and $\sin\dfrac{\theta}{4}=\sqrt{\dfrac{1-\frac{5}{7}}{2}}=\sqrt{\dfrac{1}{7}}$, so $\tan\dfrac{\theta}{4}=\dfrac{\sqrt{\frac{1}{7}}}{\sqrt{\frac{6}{7}}}=\dfrac{1}{\sqrt{6}}$.

(c) Consider the boulder as a sphere of radius R centred at the origin. The entire boulder is formed as a volume of revolution, rotating the curve $y^2=R^2-x^2$ about the y-axis.

The volume underwater is found by integrating from $-R$ to $-R\cos\phi$. We need the proportion this is of the total volume, $\dfrac{4\pi R^3}{3}$.

$$\begin{aligned}
P(\phi)&=\frac{3}{4\pi R^3}\int_{-R}^{-R\cos\phi}\pi\left(R^2-x^2\right)\mathrm{d}x\\
&=\frac{3}{4R^3}\left[R^2x-\tfrac{1}{3}x^3\right]_{-R}^{-R\cos\phi}\\
&=\frac{3}{4R^3}\left(R^2(-R\cos\phi)-\tfrac{1}{3}\left(-R\cos\phi\right)^3-R^3+\tfrac{1}{3}R^3\right)\\
&=\tfrac{3}{4}\left(-\cos\phi+\tfrac{1}{3}\cos^3\phi-1+\tfrac{1}{3}\right)\\
&=\tfrac{1}{4}\left(\cos^3\phi-3\cos\phi-2\right)\\
&=\tfrac{1}{4}(1-\cos\phi)^2(2+\cos\phi)
\end{aligned}$$

Note that, as we would expect:

$$\begin{aligned}
P(0)&=\tfrac{1}{4}(1-1)^2(2+1)=0\\
P(\tfrac{\pi}{2})&=\tfrac{1}{4}(1-0)^2(2+0)=\tfrac{1}{2}\\
P(\pi)&=\tfrac{1}{4}(1-(-1))^2(2+(-1))=1
\end{aligned}$$

[Despite the usual insistence to use radians in calculus problems, it does not matter if an answer uses angles, because the $\cos\phi$ term is not integrated.]

QUESTION FOUR SOLUTION

(a) [We hope to see diagrams supporting answers for this question.]

The horizontal tangent lines are $y = r$ and $y = -r$.

By symmetry the other lines are $y = \pm mx$. Now find *m*, by first substituting for *y* in either circle.

$$r^2 - y^2 = (x-a)^2$$
$$r^2 - m^2x^2 = x^2 - 2ax + a^2$$
$$0 = (1+m^2)x^2 - 2ax + (a^2 - r^2)$$

The roots of this equation give intersections of the line with the circle; we want a unique solution, so

$$4a^2 - 4(1+m^2)(a^2-r^2) = 0$$
$$1 + m^2 = \frac{a^2}{a^2 - r^2}$$
$$m^2 = \frac{a^2}{a^2-r^2} - 1 = \frac{a^2 - \left(a^2 - r^2\right)}{a^2 - r^2} = \frac{r^2}{a^2-r^2}$$
$$m = \frac{\pm r}{\sqrt{a^2 - r^2}}$$

So the tangent lines are $y = \dfrac{\pm rx}{\sqrt{a^2-r^2}}$.

Alternatively, if the angle between the tangent line and the *x*-axis is θ, then we find $\tan\theta = \dfrac{r}{\sqrt{a^2-r^2}}$.

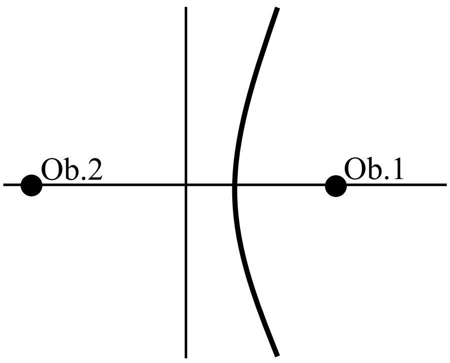
(b) $$\frac{dB}{dx} = I\left(\frac{-2}{x^3} + \frac{2}{(d-x)^3}\right) = 2I\left(\frac{1}{(d-x)^3} - \frac{1}{x^3}\right) = 2I\,\frac{x^3 - (d-x)^3}{x^3(d-x)^3} = 2I\,\frac{(2x-d)(x^2 - dx + d^2)}{x^3(d-x)^3} = 0$$

The only real root is $x = d/2$; this gives the two points equidistant from the foci, the **points on the minor axis**. The function $B(x)$ is decreasing for $x < d/2$ and increasing for $x > d/2$.

(c) For simplicity, choose a coordinate system with the observers positioned 1.5 kilometres either side of the origin on the x-axis. At a point (x, y), the difference between the distance to the points is $D = \sqrt{y^2 + (x-1.5)^2} - \sqrt{y^2 + (x+1.5)^2}$. Since sounds travels at 340 m/s, the difference in distances in 1.5 seconds is 0.51 kilometres.

$$0.51 = \sqrt{y^2 + x^2 - 3x + 2.25} - \sqrt{y^2 + x^2 + 3x + 2.25}$$
$$0.2601 = y^2 + x^2 - 3x + 2.25 + y^2 + x^2 + 3x + 2.25 - 2\sqrt{(y^2 + x^2 - 3x + 2.25)(y^2 + x^2 + 3x + 2.25)}$$
$$x^2 + y^2 + 2.11995 = \sqrt{(y^2 + x^2 - 3x + 2.25)(y^2 + x^2 + 3x + 2.25)}$$
$$(x^2 + y^2 + 2.11995)^2 = (y^2 + x^2 - 3x + 2.25)(y^2 + x^2 + 3x + 2.25)$$
$$x^4 + y^4 + 2x^2y^2 + 4.2399x^2 + 4.2399y^2 + 4.4941880025 = x^4 + y^4 + 2x^2y^2 - 4.5x^2 + 4.5y^2 + 5.0625$$
$$8.7399x^2 - 0.2601y^2 = 0.5683119975$$

It is **one side** of this hyperbola; only the points where $x > 0$ are closer to observer 1. This hyperbola is quite 'straight'; the asymptotes are approximately $y = \pm 5.8x$.



Alternatively, notice that the situation describes a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$2a = 0.34 \times 1.5 = 0.51$ since this is the difference in distance to the observers.

Since $c^2 = a^2 + b^2$ and we have $2c = 3$, $b^2 = 1.5^2 - 0.255^2 = 2.184975$.

The hyperbola is $\frac{x^2}{0.065025} - \frac{y^2}{2.14495} = 1$.

QUESTION FIVE SOLUTION

(a) First we find the value of $\Gamma(\frac{1}{2})$ and work from there:

$$\Gamma(\tfrac{1}{2})\Gamma(1-\tfrac{1}{2}) = \frac{\pi}{\sin\frac{1}{2}\pi}$$

$$\Gamma(\tfrac{1}{2})^2 = \pi$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{3}{4}\sqrt{\pi}$$

(b) The equation has real coefficients, so $\bar{c} = \sqrt{2} - \sqrt{3}\mathrm{i}$ is also a root. Since all the terms are even powers of x, so are $-c = -\sqrt{2} - \sqrt{3}\mathrm{i}$ and $-\bar{c} = -\sqrt{2} + \sqrt{3}\mathrm{i}$.

These roots give factors of the polynomial

$$\left(x-\sqrt{2}-\sqrt{3}\mathrm{i}\right)\left(x-\sqrt{2}+\sqrt{3}\mathrm{i}\right)\left(x+\sqrt{2}+\sqrt{3}\mathrm{i}\right)\left(x+\sqrt{2}-\sqrt{3}\mathrm{i}\right)$$
$$=\left(x^2-2\sqrt{2}x+5\right)\left(x^2+2\sqrt{2}x+5\right) = x^4+2x^2+25$$

By observation, we find that $(x^4+2x^2+25)(x^2-k)=0$ is a factorisation of the original equation.

The roots are $\bar{c}, -c, -\bar{c}, \sqrt{k}$ and $-\sqrt{k}$.

It is possible to find the last two roots by inspection, by substituting $x^2 = k$.

(c) Some quick geometry gives that the distance between two roots is $\sqrt{2}$ and the centre is at $C = a + b\mathrm{i} = \left(1+\frac{1}{\sqrt{2}}\right)+\left(1+\frac{1}{\sqrt{2}}\right)\mathrm{i}$.

Shifting the points to be centred on the origin, the new points are the roots of $x^8 + r^8 = 0$.

From the diagram we find $r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+1+\sqrt{2}+\frac{1}{2}} = \sqrt{2+\sqrt{2}}$

$$x^8 + \left(2+\sqrt{2}\right)^4 = 0$$

Now shift the centre from the origin to $C = \left(1+\frac{1}{\sqrt{2}}\right)+\left(1+\frac{1}{\sqrt{2}}\right)\mathrm{i}$.

$$p(z) = \left(z-(a+b\mathrm{i})\right)^8 + r^8 = \left(z-\left(1+\tfrac{1}{\sqrt{2}}\right)-\left(1+\tfrac{1}{\sqrt{2}}\right)\mathrm{i}\right)^8 + \left(2+\sqrt{2}\right)^4.$$

In the original question, as well as the obvious $n = 8$, we have $a + b\mathrm{i} = \left(1+\frac{1}{\sqrt{2}}\right)+\left(1+\frac{1}{\sqrt{2}}\right)i = (1+\mathrm{i})\left(1+\frac{1}{\sqrt{2}}\right)$ and $q = \left(2+\sqrt{2}\right)^4 = 68 + 48\sqrt{2}$.