Assessment Schedule – 2012

Scholarship Mathematics with Calculus (93202)

Evidence Statement

9 marks possible in each question, to a maximum of 8

Minor Error Ignored (MEI) is typically a single character inserted, omitted, or incorrect

QUESTION ONE SOLUTIONS

(a) (i)

$$\left(\sqrt[3]{a+b}+\sqrt[3]{a-b}\right)^3=\left(\sqrt[3]{a+b}\right)^3+3\left(\sqrt[3]{a+b}\right)^2\sqrt[3]{a-b}+3\sqrt[3]{a+b}\left(\sqrt[3]{a-b}\right)^2+\left(\sqrt[3]{a-b}\right)^3$$
$$=a+b+3\sqrt[3]{a+b}\sqrt[3]{a-b}\left(\sqrt[3]{a-b}+\sqrt[3]{a+b}\right)+a-b$$
$$=2a+3\sqrt[3]{a^2-b^2}\left(\sqrt[3]{a-b}+\sqrt[3]{a+b}\right)$$

(ii)

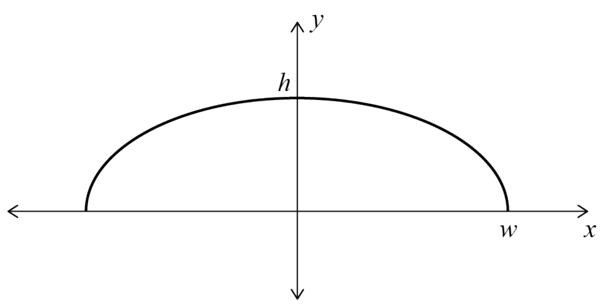
$$\rho^3=\left(\sqrt[3]{\tfrac{1}{2}+\tfrac{1}{6}\sqrt{\tfrac{23}{3}}}+\sqrt[3]{\tfrac{1}{2}-\tfrac{1}{6}\sqrt{\tfrac{23}{3}}}\right)^3$$
$$=2\times\tfrac{1}{2}+3\sqrt[3]{\left(\tfrac{1}{2}\right)^2-\tfrac{1}{36}\tfrac{23}{3}}\left(\sqrt[3]{\tfrac{1}{2}+\tfrac{1}{6}\sqrt{\tfrac{23}{3}}}+\sqrt[3]{\tfrac{1}{2}-\tfrac{1}{6}\sqrt{\tfrac{23}{3}}}\right)$$
$$=1+3\sqrt[3]{\tfrac{27}{108}-\tfrac{23}{108}}\rho$$
$$=1+3\sqrt[3]{\tfrac{1}{27}}\rho$$
$$=1+3\times\tfrac{1}{3}\rho$$
$$=1+\rho$$

(b) Ellipse: $\dfrac{x^2}{w^2}+\dfrac{y^2}{h^2}=1$ rearranges to $y^2=\dfrac{h^2}{w^2}\left(w^2-x^2\right)$

Parabola: $y=kx^2+h$, but when $y=0,\quad x=w\quad \Rightarrow k=\dfrac{-h}{w^2}$

Parabaloid has volume $\dfrac{16}{15}\pi h^2 w$

Ratio:



$$\frac{\text{volume of ellipse}}{\text{volume of parabola}}=\frac{2\pi\dfrac{h^2}{w^2}\displaystyle\int_0^w\left(w^2-x^2\right)\mathrm{d}x}{2\pi\dfrac{h^2}{w^4}\displaystyle\int_0^w\left(w^4-2w^2x^2+x^4\right)\mathrm{d}x}$$
$$=\frac{w^2\left[w^2x-\dfrac{x^3}{3}\right]_0^w}{\left[w^4x-\dfrac{2w^2x^3}{3}+\dfrac{x^5}{5}\right]_0^w}=\frac{\tfrac{2}{3}}{\tfrac{8}{15}}=\frac{5}{4}$$

ONE(a)(i)

1	expand cubic $(x+y)^3=x^3+3x^2y+3xy^2+y^3$	need not use binomial expansion, can also do algebraically
1	simplify to required expression	not sufficient to write the given answer; but acceptable to work from both ends to a common statement
1	correct mathematical expressions need at least one of above marks	uses equals in all appropriate places, and no inappropriate places does not use $\Rightarrow$ to mean =

ONE(a)(ii)

1	apply the fact from ONE(a)(i) on ρ^3	need to do more than write $\left(\sqrt[3]{x+y}+\sqrt[3]{x-y}\right)^3$
1	get as far as factor: $3\times\sqrt[3]{a^2-b^2}=3\times\sqrt[3]{\tfrac{1}{27}}$	
to 3	final result showing $=1+\rho$	not sufficient to write the given answer

ONE(b)

1	equation of correct parabola $y=\dfrac{-h}{w^2}x^2+h$	$y=\dfrac{h}{w^2}x^2-h$ is also acceptable
1	volume of ellipsoid $2\pi\dfrac{h^2}{w^2}\left(w^3-\dfrac{w^3}{3}\right)=\dfrac{4}{3}\pi h^2 w$	some parameters may have been eliminated early if the ratio is formed in the working early
1	final ratio 5:4, or $\dfrac{5}{4}$, or 1.25, with consistency for an answer having no h or w	As long as the ratio is described, finding the reciprocal is acceptable

Working with different parameters for h and w is acceptable, if consistent.

Working with arbitrarily chosen numerical parameters for h and w is not acceptable; two mark maximum.

QUESTION TWO SOLUTIONS

(a)

$$nx^{n-1} = \frac{\mathrm{d}y}{\mathrm{d}x} n\sec^2(ny)$$

$$x^{n-1} = \frac{\mathrm{d}y}{\mathrm{d}x}\left[1+\tan^2(ny)\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^{n-1}}{1+x^{2n}}$$

Without using the trigonometric identity, $\dfrac{dy}{dx} = \dfrac{x^{n-1}}{\sec^2\left(\tan^{-1} x^n\right)}$

(b)

$$2Bx + 2Cy\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{Bx}{Cy}$$

For the orthogonal family of curves, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Cy}{Bx}$. Using separation of variables

$$\frac{1}{y}\mathrm{d}y = \frac{C}{B}\frac{1}{x}\mathrm{d}x$$

$$\ln|y| = \frac{C}{B}\ln|x| + c$$

$$y = e^{\frac{C}{B}\ln x + c}$$

$$\ln|y| = \ln\left|kx^{C/B}\right|$$

$$y = kx^{C/B}$$

(c) In each of the examples given, the rate of change either increases or decreases proportionally to the current amount. This can be modelled as $\frac{\mathrm{d}A}{\mathrm{d}t} \propto A$ and hence, the differential equation $\frac{\mathrm{d}A}{\mathrm{d}t} = \mathrm{k}A$ where k is a constant.

Using integration $\frac{\mathrm{d}A}{\mathrm{d}t} = \mathrm{k}A \Rightarrow \frac{\mathrm{d}A}{A} = \mathrm{k}\,\mathrm{d}t \Rightarrow \int\frac{\mathrm{d}A}{A} = \int \mathrm{k}\,\mathrm{d}t$

$\ln\left|\frac{A}{A_0}\right| = \mathrm{k}t \Rightarrow A = A_0\mathrm{e}^{\mathrm{k}t}$. This produces the model where A_0 is the current value (or initial value) and k will be a positive constant for an increase and negative for a decrease.

In many cases, the exponential model can only be used in a limited time frame as the conditions might change, eg in the case of the spread of an epidemic.

TWO(a)

1	correct implicit differentiation, both sides	need the $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the right place
1	use trig identity, e.g. $\sec^2 x = 1+\tan^2 x$, to get to $\tan^2 x$	
to 3	rearrange to $\frac{\mathrm{d}y}{\mathrm{d}x} =$	not containing any occurrences of y

Note that $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x)) = \frac{1}{x^2+1}$ could be used for the first two marks

TWO(b)

1	$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2Bx}{2Cy} = -\frac{Bx}{Cy}$	no negative sign is MEI
1	orthogonality – intersecting at right angles means $m_\perp = \frac{-1}{m}$	any evidence of this fact
1	$y = kx^{C/B}$	or an equivalent form, even $y = e^{\frac{C}{B}\ln x + c}$

TWO(c) The first statement, one of the calculus statements, and any other +

1*	rate of change is proportional to amount	could write the differential equation $\frac{\mathrm{d}A}{\mathrm{d}t} = kA$
1+	integrate differential equation to get $A(t) = A_0e^{kt}$	use of calculus required, related to the proportionality above can get marks for **both**
1+	differentiate $A(t)$ to show proportionality, eg $\frac{\mathrm{d}A}{\mathrm{d}t} = kA$	
+	describe role of A_0 in model	initial value at $t = 0$
+	describe role of k in model	positive for growth, negative for decay, fast and slow by magnitude
+	limitations of model	Malthusian growth for only a limited time; extrapolation limited
+	horizontal asymptote $A = 0$	especially in relation to decay to zero
+	$\ln A = \ln A_0 + kt$	linear form of log-graph
+	other statements	as approved by Panel Leader

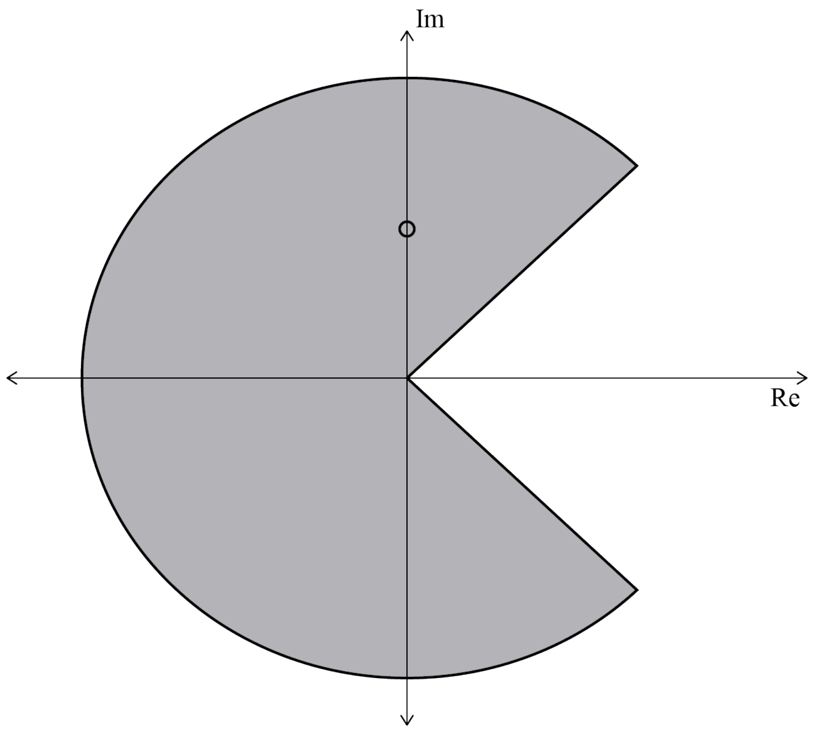
QUESTION THREE SOLUTIONS

(a) (i) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\cos x\right)=\cos x-x\sin x$

Hence by observation, $\int x\sin x\,\mathrm{d}x=\sin x-x\cos x+c$

(ii) $\int_0^{n\pi} x\sin x\,\mathrm{d}x=\left[\sin x-x\cos x\right]_0^{n\pi}=\sin\left(n\pi\right)-n\pi\cos\left(n\pi\right)=-n\pi\cos\left(n\pi\right)=(-1)^{n+1}n\pi$

(b) Diagram should also be labelled, or otherwise described, to show the radius is also 1, and the 'eye' is at $0+0.512\mathrm{i}$. The 'mouth' is open to an angle of $\frac{\pi}{2}$, with one edge at an angle of $\frac{\pi}{4}$



THREE(a)(i)

1	$\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x-x\sin x$	
1	hence $\int x\sin x\,\mathrm{d}x=\sin x-x\cos x$	by observation from the result
1	$+c$	the constant must be there

THREE(a)(ii)

1	$=\sin(n\pi)-n\pi\cos(n\pi)$	find definite integral
1	$\sin(n\pi)=0$	note this
to 3	$(-1)^{n+1}n\pi=\begin{cases}n\pi \text{ if } n \text{ is odd}\\ -n\pi \text{ if } n \text{ is even}\end{cases}$	either form, with specific cases: $\pm n\pi$ is not sufficient

THREE(b)

1	pacman shape: sector between 180 and 360 degrees	mouth open to the right
1	correct z^3 for the eye	close to $0.5\mathrm{i}$, on imaginary axis
1	pacman mouth open at 90 degrees, 45 degrees above and below	either labelled with angles in diagram, or angles calculated in working elsewhere

QUESTION FOUR SOLUTIONS

(a)

$f(x) = \log_m x + \log_x m = \dfrac{\ln x}{\ln m} + \dfrac{\ln m}{\ln x}$	using identities for logarithms
$\dfrac{df}{dx} = \dfrac{1}{\ln m}\dfrac{1}{x} + \ln m \dfrac{1}{x}\dfrac{-1}{(\ln x)^2}$	using the chain (or quotient) rule
$\dfrac{df}{dx} = 0$	setting derivative to zero to find critical point
$\dfrac{1}{\ln m}\dfrac{1}{x} = \dfrac{1}{x}\dfrac{\ln m}{(\ln x)^2}$	rearranging to get in terms of x
$(\ln x)^2 = (\ln m)^2$	note that $x \neq 0$
$\ln x = \pm \ln m$	positive and negative solutions are possible,
$x = m$ or $x = \dfrac{1}{m}$	but we rule out the second as $m, x > 1$

Minimum value is $\log_m m + \log_m m = 2$

FOUR(a)		
1	find $\dfrac{df}{dx} = \dfrac{1}{x \ln m} - \dfrac{\ln m}{x(\ln x)^2}$	any form
to 2	minimum value is at $f(m) = 2$	**evidence** required (minimum test not required)
+1	"clearly explain the steps of your working" explains correct and valid steps	need not be as complete as shown, but must explain at least TWO key decisions made

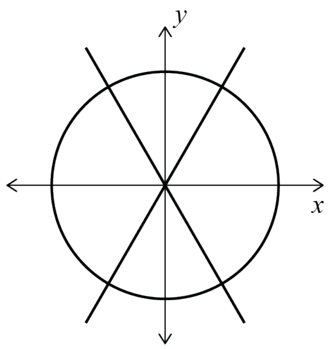
(b) Substituting, we find $y^2 - q^2x^2$ is a factor:

$$y^4 + (1-q^2)x^2y^2 - q^2x^4 + q^2x^2 - y^2 = 0$$
$$q^4x^4 + (1-q^2)q^2x^4 - q^2x^4 + q^2x^2 - q^2x^2 = 0$$
$$q^4x^4 + q^2x^4 - q^4x^4 - q^2x^4 + q^2x^2 - q^2x^2 = 0$$
$$0 = 0$$

So we can factorise:

$$y^4 + (1-q^2)x^2y^2 - q^2x^4 + q^2x^2 - y^2 = 0$$
$$(y^2 - q^2x^2)(y^2 + x^2 - 1) = 0$$
$$(y+qx)(y-qx)(y^2 + x^2 - 1) = 0$$

This gives the lines $y = \pm qx$ and the unit circle $x^2 + y^2 = 1$.



FOUR(b)		
1	both $y = \pm qx$ as equations of lines	need not be drawn, could be labelled in diagram
1	factorise to find other factor $x^2 + y^2 - 1$	need not recognise as a circle
1	diagram with circle and lines crossing at centre	lines of any slopes $\pm q$ with reflection symmetry. Axes not required. Labelling radius $r = 1$ not required.

(c)

$$\tan 4x\left(\tan^2 x-2\tan x-1\right)\left(\tan^2 x+2\tan x-1\right)$$
$$=\frac{2\tan 2x}{1-\tan^2 2x}\left(\tan^2 x-2\tan x-1\right)\left(\tan^2 x+2\tan x-1\right)$$
$$=\frac{2\tan 2x}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}\left(\tan^2 x-2\tan x-1\right)\left(\tan^2 x+2\tan x-1\right)$$
$$=\frac{2\tan 2x\left(1-\tan^2 x\right)^2}{\left(1-\tan^2 x\right)^2-4\tan^2 x}\left(\tan^2 x-2\tan x-1\right)\left(\tan^2 x+2\tan x-1\right)$$
$$=\frac{2\tan 2x\left(1-\tan^2 x\right)^2}{\left(1-\tan^2 x-2\tan x\right)\left(1-\tan^2 x+2\tan x\right)}\left(\tan^2 x-2\tan x-1\right)\left(\tan^2 x+2\tan x-1\right)$$
$$=2\tan 2x\left(1-\tan^2 x\right)^2=2\tan 2x\left(\tan^2 x-1\right)^2=2\tan 2x\left(\tan x-1\right)^2\left(\tan x+1\right)^2$$

OR

Using $\tan 4x=\frac{2\tan 2x}{1-\tan^2 2x}\Rightarrow 2\tan 2x=\tan 4x\left(1-\tan^2 2x\right)$

$$2\tan 2x\left(\tan x-1\right)^2\left(\tan x+1\right)^2=\tan 4x\left(1-\tan^2 2x\right)\left(\tan x-1\right)^2\left(\tan x+1\right)^2$$
$$=\tan 4x\left(1-\tan^2 2x\right)\left(\tan^2 x-1\right)^2$$
$$=\tan 4x\left(1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2\right)\left(\tan^2 x-1\right)^2$$
$$=\tan 4x\left(\frac{\left(1-\tan^2 x\right)^2-4\tan^2 x}{\left(1-\tan^2 x\right)^2}\right)\left(\tan^2 x-1\right)^2$$
$$=\tan 4x\left(\left(1-\tan^2 x\right)^2-4\tan^2 x\right)$$
$$=\tan 4x\left(1-\tan^2 x-2\tan x\right)\left(1-\tan^2 x+2\tan x\right)$$
$$=\tan 4x\left(\tan^2 x+2\tan x-1\right)\left(\tan^2 x-2\tan x-1\right)$$

FOUR(c)

Because there are different approaches possible, the marks are for two skills and then for completing the whole identity.

1	correct use of double angle formula for $\tan(4x)$	note: not if used on the LHS, this is unhelpful
1	RHS expanded to contain at most one $\tan 2x$ term	
to 3	full proof of identity given	any valid proof, regardless of approach, gets full marks

Converting to $\sin(x)$ and $\cos(x)$ is unlikely to lead to fruition.

Also note that that the expressions are equivalent to $\frac{8\tan^2 x}{\tan 2x}$

QUESTION FIVE SOLUTIONS

(a) (i) If the lines are separated by equal angles, the angles made by the lines are

$\frac{\pi}{12}+\frac{n\pi}{6}$, giving gradients $\pm\tan\frac{\pi}{12}, \pm\tan\frac{\pi}{4}, \pm\tan\frac{5\pi}{12}$

Two of the lines are $y=-x$ and $y=x$. These are true when $x+y=0$ and $x-y=0$, together giving $(x+y)(x-y)=(x^2-y^2)=0$.

Now note that $\left(\tan\frac{\pi}{12}\right)^2=\left(2-\sqrt{3}\right)^2=4+3-4\sqrt{3}=7-4\sqrt{3}$.

Similarly $\left(\tan\frac{5\pi}{12}\right)^2=\left(2+\sqrt{3}\right)^2=4+3+4\sqrt{3}=7+4\sqrt{3}$.

Also, $\left(7+4\sqrt{3}\right)\left(7-4\sqrt{3}\right)=1$.

Pairing the other lines together: $y=\left(2-\sqrt{3}\right)x$ and $y=-\left(2-\sqrt{3}\right)x$ give $\left(y-\left(2-\sqrt{3}\right)x\right)\left(y+\left(2-\sqrt{3}\right)x\right)=\left(y^2+\left(7-4\sqrt{3}\right)x^2\right)=0$.

All together, we get $\left(x^2-y^2\right)\left(x^2-\left(7-4\sqrt{3}\right)y^2\right)\left(x^2-\left(7+4\sqrt{3}\right)y^2\right)=0$.

OR, Starting from the given form:

$x^2-y^2=0$ gives the lines $y=\pm x$ with gradients $\pm\tan\frac{\pi}{4}$

$x^2-\left(7-4\sqrt{3}\right)y^2=0$ gives the lines $y=\frac{\pm x}{2-\sqrt{3}}=\pm\left(2+\sqrt{3}\right)x$ with gradients $y=\pm\tan\frac{5\pi}{12}$

$x^2-\left(7+4\sqrt{3}\right)y^2=0$ gives the lines $y=\frac{\pm x}{2+\sqrt{3}}=\pm\left(2-\sqrt{3}\right)x$ with gradients $y=\pm\tan\frac{\pi}{12}$

(ii) For a hyperbola in the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the asymptotes of the hyperbola must be steeper than all of the lines for the hyperbola to pass above.

The steepest gradients of the lines are $m=\pm\sqrt{7+4\sqrt{3}}$

So asymptotes of $y=\pm\frac{b}{a}x$ would need to have satisfy the condition $\left(\frac{b}{a}\right)^2>7+4\sqrt{3}$ (≈ 13.9282) and equation of the hyperbola could be $\frac{y^2}{7+4\sqrt{3}}-\frac{x^2}{1}=1$ or $\frac{y^2}{1}-\frac{x^2}{7-4\sqrt{3}}=1$

<u>Note:</u> For hyperbola in format $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, asymptotes have gradients such that $\frac{a}{b}>\sqrt{7-4\sqrt{3}}$ and equation could be $\frac{x^2}{7+4\sqrt{3}}-\frac{y^2}{1}=1$ or $\frac{x^2}{1}-\frac{y^2}{7-4\sqrt{3}}=1$.

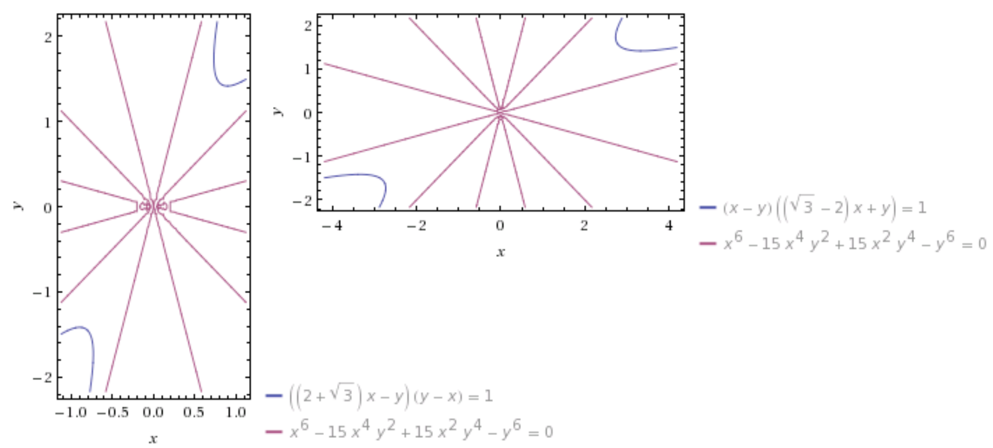
FIVE(a)(i)

1	lines $y=\pm x$	$x^2=y^2$ is not sufficient
1	$\left(2+\sqrt{3}\right)^2=7+4\sqrt{3}$ $\left(2-\sqrt{3}\right)^2=7-4\sqrt{3}$	both must be shown (expect to see separately, but allow $\left(2\pm\sqrt{3}\right)^2=7\pm4\sqrt{3}$
1	other paired lines form difference of two squares (x^2-a^2) as an equation $=0$	inappropriate pairing not a minor error; must be $(x-a)(x+a)$

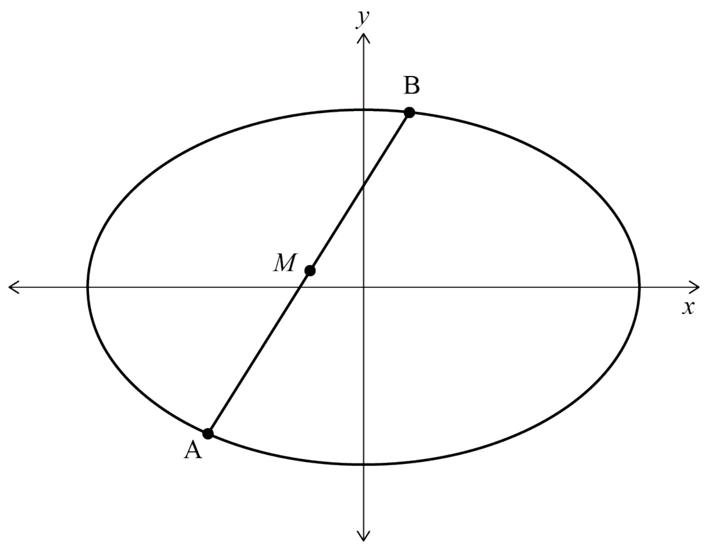
FIVE(a)(ii)

1	describe suitable asymptotes of their hyperbola asymptotes can be implied from the equation or diagram	if in steeper than form, must be steeper than $y=\pm\sqrt{7+4\sqrt{3}}x=\pm\left(2+\sqrt{3}\right)x$ if in shallower than form, must be shallower than $y=\pm\left(2-\sqrt{3}\right)x$
1	construct hyperbola in appropriate form	A misplaced minus sign changes a correct hyperbola to another which crosses all the lines – this is not a minor error
1	*giving reasons for your answer.* "The hyperbola has to sit between these two lines:"	Explanation of what the asymptotes of the hyperbola are in relation to the lines.

It is possible to put a hyperbola through any adjacent pair of lines as asymptotes, but we do not expect to see these answers. Such answers contain an *xy* term when expanded.



(b) Consider the general equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a set of chords with gradient m. One such chord $y=mx+c$ cuts the hyperbola at A and B.



The x coordinates of A and B are the solutions of the quadratic $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

Rearranging the equation: $\left(b^2+a^2m^2\right)x^2+\left(2a^2cm\right)x+a^2\left(c^2-b^2\right)=0$ (1)

At the midpoint $M(X,Y)$ of AB

$$X=\frac{1}{2}\left(x_A+x_B\right)=\frac{1}{2}\left(\text{sum of the roots}\right)=\frac{1}{2}\times\frac{-2a^2cm}{b^2+a^2m^2}=\frac{-a^2cm}{b^2+a^2m^2} \qquad (2)$$

M is a point on the line $y=mx+c$, ie $Y=mX+c$

Eliminate c from equation (2): $c=\frac{\left(b^2+a^2m^2\right)X}{-a^2m}$

Hence $Y=mX-\frac{\left(b^2+a^2m^2\right)X}{a^2m}\Rightarrow a^2mY=a^2m^2X-b^2X-a^2m^2X$

$b^2X+a^2mY=0$ is the locus of midpoints of the chords, and is a straight line passing through the origin.

FIVE(b)

Diagrams to support the answer are expected, to illustrate the uses of the new variables introduced.

1	quadratic for coordinates of intersection points $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$	substitution $y=mx+c$ or similar must be made for a quadratic in x only
1*	half the sum of roots from the expanded quadratic: $X=\frac{-a^2cm}{b^2+a^2m^2}$ MEI re-using x again	alternatively, it is possible to find the roots using the quadratic formula, and take their average without realising that the average of $\frac{-B\pm\sqrt{B^2-4AC}}{2A}$ is $\frac{-B}{2A}$
+	or, in a partial attempt, noting from diagram or otherwise that line $Y=kX$ goes through the origin	mark allocated when 1* above is **not** reached; could be in various forms
to 3	eliminate c to find any linear equation in X and Y: $Y=mX-\frac{b^2+a^2m^2}{a^2m}X$ $=\left(m-\frac{b^2+a^2m^2}{a^2m}\right)X$	without eliminating the variable c, we do not have an equation for a line