NZEST SCHOLARSHIP EXAMINATION 1994

MATHEMATICS WITH CALCULUS

Time allowed: THREE hours

Ten minutes extra are allowed for reading this paper

Notes for Candidates

1. This examination consists of SIX questions, each worth 20 marks
2. Please attempt **ALL SIX** questions
3. You will get credit for every question (or part question) you do
4. Full marks can be obtained from five questions
5. Please show all working
6. One sheet of graph paper is available for use if required - if you use the graph paper, please tie it into your answer book
7. Calculators may be used
8. Tables of formulae are attached at the back of the question paper

Question 1

(a) By writing out series expansions for e^x and also for e^{-x}, (refer tables of formulae), obtain series expansions for :

(i) $\frac{e^x - e^{-x}}{2}$, (also called sinh(x)), and

(ii) $\frac{e^x + e^{-x}}{2}$, (also called cosh(x)). [4]

Show that the derivative of sinh(x) is cosh(x). [2]

Find the derivative of cosh(x). [1]

(b) Using the substitution method, evaluate the integral :

$$\int_0^5 (2x-5)\sqrt{2x+5}\ dx$$

[3]

Explain why the method you have just used cannot be used to find the following :

$$\int_0^5 (2x+5)\sqrt{2x-5}\ dx$$

[1]

(c) Given that y = 4 when x = 0, solve the differential equation

$$\frac{dy}{dx} = y + 1.$$

[4]

(d) Assuming the well-known result

$$\binom{r}{s} + \binom{r}{s+1} = \binom{r+1}{s+1},$$

prove by induction on n that

$$\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \ldots + \binom{5+n}{n} = \binom{6+n}{n}.$$

[5]

CONTINUED

Question 2

Sets of complex numbers can be represented as points on an Argand diagram. Frequently we use a parameter, t, which is a variable real number, as a way of describing a certain set of complex numbers.

(a) Let

$$z(t) = 1 + t + (3 + t)\,i, \qquad t \in \boldsymbol{R}. \qquad (1)$$

Evaluate $z(t)$ for $t = 0, 1, 2, 3$ and 4. Plot these complex numbers as points on an Argand diagram. Complete the diagram to show the entire locus of equation (1). [3]

(b) A better way to deal with equation (1) is to separate the real and imaginary parts, obtaining the simultaneous equations

$$x = 1 + t$$
$$y = 3 + t\,,$$

where x represents the real part of z, and y represents the imaginary part of z. *Eliminate the parameter t to obtain an equation in x and y only.* [2]

(c) Let

$$z(t) = (t + i)^2, \qquad t \in \boldsymbol{R}. \qquad (2)$$

Using a similar method to that outlined in (b), or otherwise, prove that the locus of equation (2) forms a parabola. Sketch a diagram to show this parabola. [5]

(d) Let

$$z(t) = \frac{t - i}{t + i}, \qquad t \in \boldsymbol{R}. \qquad (3)$$

By eliminating the parameter t, show that the locus of equation (3) on an Argand diagram can be represented by the equation $x^2 + y^2 = 1$, with $x \neq 1$. [5]

(e) Let

$$z(t) = \frac{t + ki}{t + i}, \qquad t \in \boldsymbol{R},\ k \in \boldsymbol{R},\ k \neq 1. \qquad (4)$$

Prove that the locus of equation (4) will always lie on a circle, for any given value for k except 1. Give the coordinates of the centre, and also the radius, as a function of k. [5]

Question 3

At the Waipa sawmill new technology has just been installed to speed up production and to reduce waste. Pine logs are fed into the mill in their rough state, which is approximately cylindrical. As each *log* enters the mill, a laser device measures its *diameter* at 1-metre intervals. (Refer figure 2.) From these measurements a computer calculates the approximate volume of the log, and the best plan for slicing it up into *slabs*. Within a few seconds this decision has been made, and the log is cut length-wise into slabs. Each of these slabs is then passed under a second laser device, which measures its *width* at 1-metre intervals. (Refer figure 1.) From these measurements the approximate surface area of each slab is calculated, and a decision is made as to how best to cut it into *planks* of sawn timber.

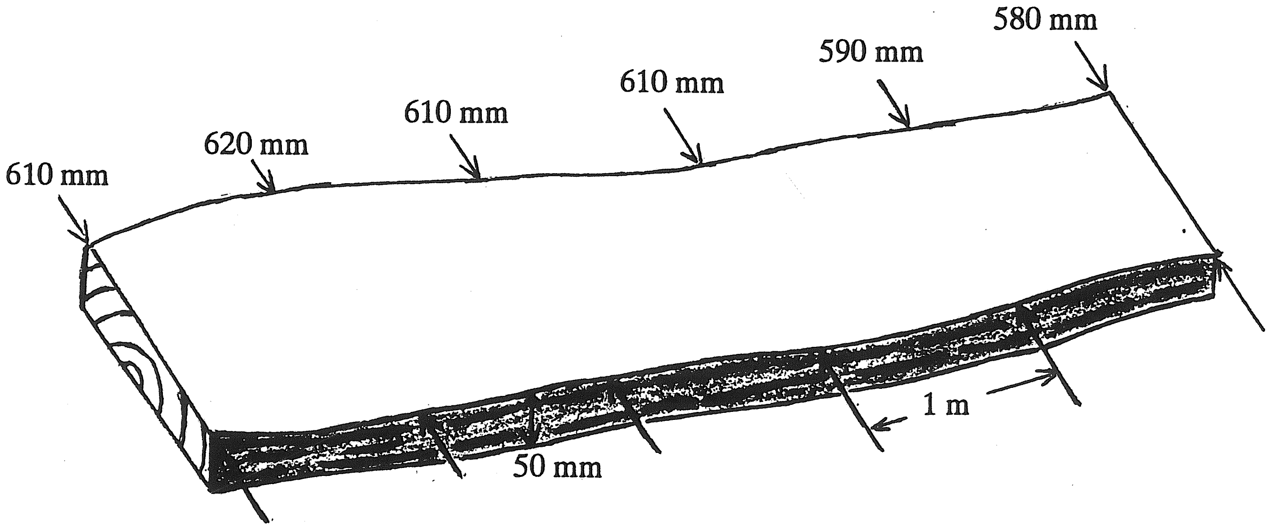

Figure 1 : A slab after it has been measured for width.

(a) Figure 1 shows a slab, 50mm thick, and 5 metres long, along with the laser measurements of its width. Use the trapezium rule to find the approximate surface area of the top of this slab. [3]

(b) The mill has an order for a large quantity of planks 150 mm by 50 mm with a minimum length of 3 metres. How many of these planks can be cut from the slab given in figure 1 ? You may assume the slab is approximately symmetrical about its length-wise axis. Include in your answer a diagram to show exactly how this slab should be cut up. Draw two diagrams to show that you have thought of more than one way to attempt this. Although the planks have a minimum length, it is desirable to make each plank as long as possible. Indicate which of your two diagrams gives the better result. Assume any wood not cut into the desired 150 mm by 50 mm planks will be wasted. The width of a saw cut is only about 2mm, and may be ignored. [4]

CONTINUED

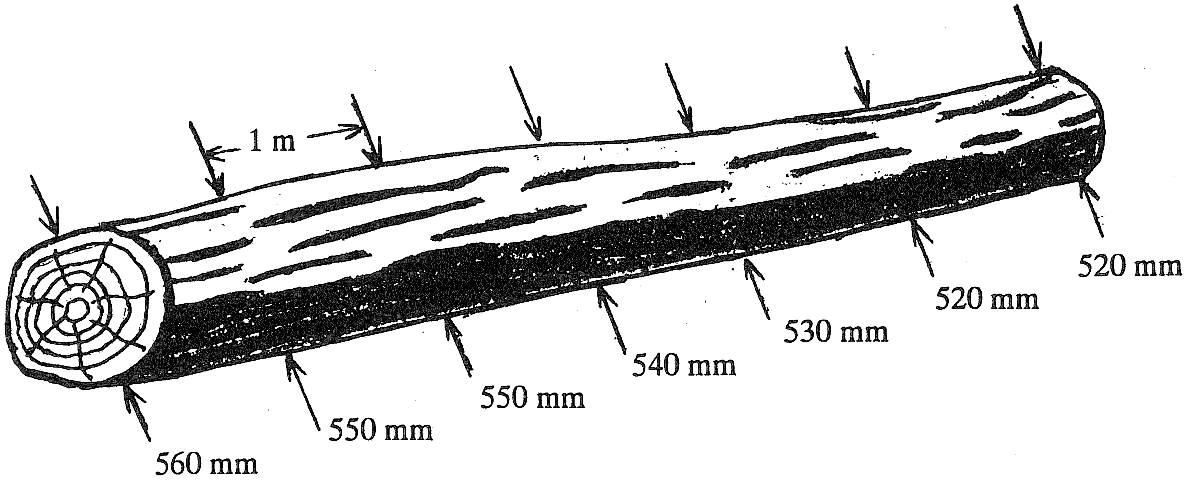

Figure 2: A pine log which has just been measured for diameter.

(c) Figure 2 shows a pine log, which is 6 metres long, as it enters the mill, along with the laser measurements of its diameter. Construct a graph showing length along the pine log on the x-axis, and the corresponding cross-sectional area on the y-axis. Hence use Simpson's rule to find the approximate volume of this log. [4]

(d) The mill has an order for timber beams 450mm by 300mm, with a minimum length of 4 metres. Is it possible to cut one of these beams out of the log shown in figure 2 ? **Either** give reasons why this is possible, including a diagram showing exactly how it could be cut from the log, **or** prove that it is impossible. You may assume the log is approximately symmetrical about its length-wise axis. [4]

(e) Another way to find the volume of a pine log is to measure only the two end diameters. We then assume that the pine log is perfectly symmetrical, and that it tapers regularly from big end to narrow end. A straight line graph is drawn to represent the radius plotted on the y-axis against length along the pine log on the x-axis. The volume is found by the calculus method of volumes of revolution. Use this method to make a revised estimate of the volume of the log shown in figure 2. Be sure to use only the two end measurements this time. [4]

(f) Compare your answers to parts (c) and (e). Which answer do you expect to be the more accurate, and why? [1]

Question 4

A large area of land on the shore of the new Lake Dunstan has been allocated for use as an outdoor education centre. At present the land is surveyed and fenced as shown by the shaded area in figure 3. Using the x- and y-axes as given, and taking the length units in kilometres, the shore of the lake follows the curve represented by $y = 1/x$.

(a) What is the total area of this land ? [3]

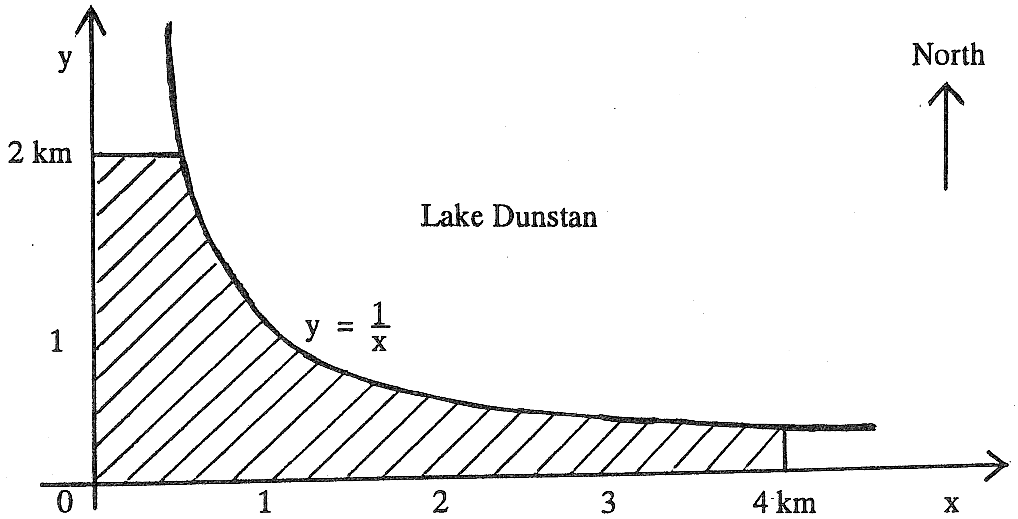

Figure 3: Map showing area to be used for an outdoor education centre.

(b) A camp is to be built on part of this land. The camp is to occupy one quarter of the land. The rest is to be fenced off, and left as a reserve of undisturbed tussockland for scientific field work. There is an existing boundary fence around the perimeter of the entire property. Determine where the new fence is to be built, given that it is to be in a North-South direction, as shown by the line AB in figure 4. [5]

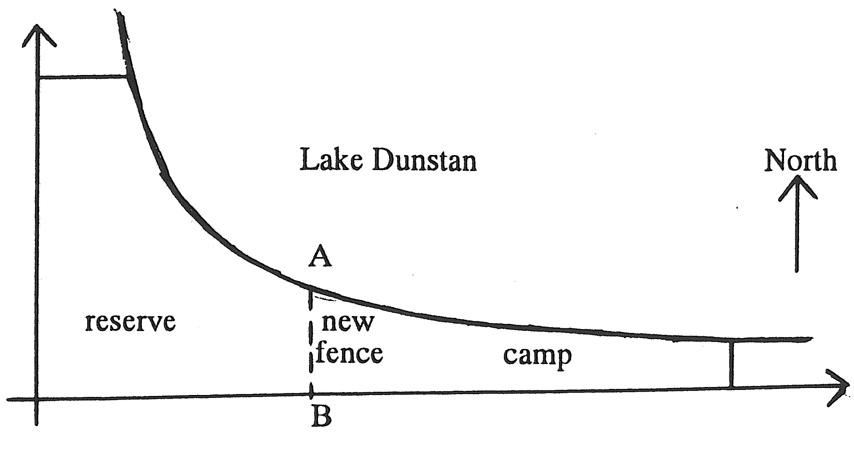

Figure 4: The area is divided by a North-South fence.

CONTINUED

(c) Suppose instead that the fence is to be placed in an East-West direction. Where should the fence be built in this case ? The camp must still occupy exactly one quarter of the area. Notice there are two possible answers here. Find them both. Do not worry if the camp loses its beach frontage on the lake. You will need to draw your own diagram for this part of the question. It may be helpful to reflect your diagram in the line $y = x$. [5]

(d) Because fencing is very expensive, the new fence is to be made as short as possible. Find the length of fence required for each of the alternatives considered in parts (b) and (c). Which of these alternatives will require the shortest fence ? [4]

(e) Another way of building the fence is to have it slanting, with one end at the South-West corner, as shown by the line CD in figure 5. Remember, the camp must still occupy exactly one quarter of the area. Prove that this method cannot give a shorter length of fence than the best answer you found in (d). (It is not necessary to find the exact location of the fence to answer this part of the question.) [3]

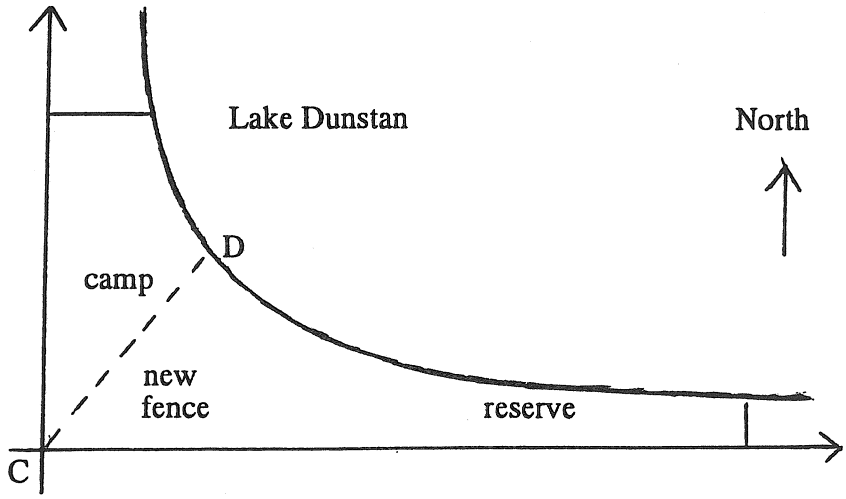

Figure 5: The area is divided by a slanting fence, with one end at the corner C.

Question 5

(a) The C.N.Z. bank offers customers an *investment account* in which a single initial deposit increases in value by $k\%$ each year, (that is, $k\%$ compound interest, payable yearly as an increase to the amount in the account). No withdrawals are allowed, until the entire amount is paid out after n years. By considering this as an example of a geometric sequence, or otherwise, show that the value of this investment account at the end of n years will be

$$A\left(1+\frac{k}{100}\right)^n \text{ dollars,}$$

where the initial deposit is A dollars, and there are no further deposits. Of course, n must be a positive integer, and k is a positive real number. [4]

(b) Prove that the value of this investment account after n years will always be greater than or equal to

$$A\left(1+\frac{nk}{100}\right) \text{ dollars.}$$

[3]

(c) Consider the binomial expression

$$\left(1+\frac{k}{100}\right)^n .$$

(i) Expand this as far as the first four terms. [2]

(ii) If the interest rate is 3% (ie $k = 3$), how large would n have to be for the third term to be greater than the second term ? [3]

(d) The bank also offers a *planned savings account* in which the customer makes a regular deposit of A dollars at the start of each year, with $k\%$ compound interest paid yearly on the entire amount in the account during the current year, as in part (a). Once again, no withdrawals are permitted until the entire amount is paid out after n years. Show that the amount obtained from this account at the end of n years will be

$$A\left(\frac{k+100}{k}\right)\left(\left(1+\frac{k}{100}\right)^n - 1\right).$$

[4]

(e) Due to inflation the currency depreciates by $d\%$ per year. This means that to find the *real value* of the amount obtained from either of the above accounts, the final amount (as given above) must be multiplied by a factor of

$$\left(1-\frac{d}{100}\right)^n .$$

Here d is a positive real number less than 100. If the interest rate, k, is chosen by the bank to be equal to the depreciation rate, d, then prove that neither the investment account (refer (a) above), nor the planned savings account (refer (d) above), will result in a real increase in the value of the customer's wealth. [4]

CONTINUED

Question 6

At the Taupo Summer Carnival it is decided to have a wind-surfing competition. A wind-surfer is a type of small sailing boat. All competitors are to use the same wind-surfer. In pre-race trials it is found that the performance of this wind-surfer depends on the angle ϕ, between the direction the wind is coming from, and the direction in which the windsurfer is sailing. Notice it cannot go directly into the wind. Refer figure 6 below. The speed, s, in kilometres per hour, of the windsurfer in a steady wind of 10 k.p.h. is given by the formula :

$$s(\phi) = 5(1 - \cos(2\phi) - \cos(\phi)), \qquad 0 \leq \phi \leq 2\pi.$$

We are assuming all competitors are equally skilful, and hence the performance of the windsurfer depends only on the choice of the angle ϕ.

(a) Find the maximum speed this wind surfer can go, and the value of ϕ for which this speed is possible. [6]

(b) If the speed drops below 0.5 k.p.h. the competitor will fall off the windsurfer. So 0.5 is the minimum speed possible in this competition. For what value of ϕ is this minimum speed achieved ? [4]

For the competition there are two long parallel lines 100 metres apart, marked on the lake with buoys. Each competitor has to sail once from any point on the starting line, to any point on the finishing line. The wind, which is still at a steady 10 k.p.h., is blowing directly from the finishing line to the starting line. Refer to figure 6, which also shows (as the dotted line AB) a possible path for a competitor.

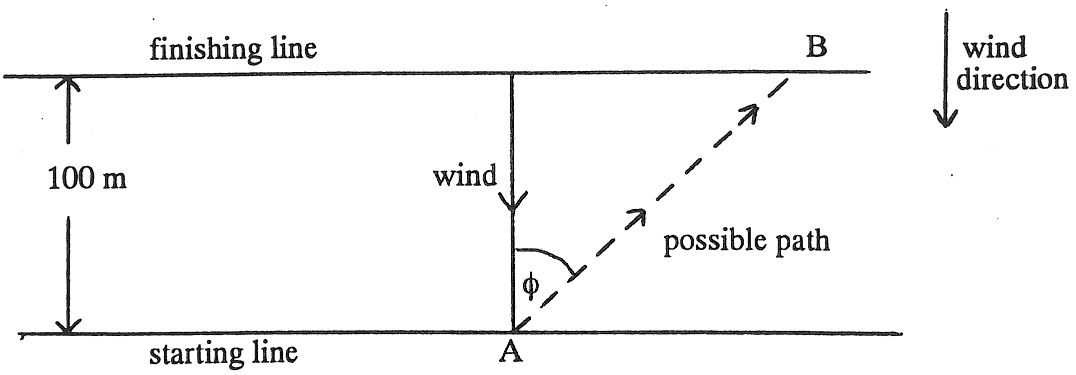

Figure 6: The water area used for the windsurfing competition.

(c) Give the *shortest distance* a competitor would have to sail so as to complete this competition without falling off the windsurfer. [3]

(d) Explain why a winning competitor is unlikely to sail at the *fastest speed* calculated in part (a). [1]

(e) Find the *shortest time* a competitor could take to complete the competition, and give the value of ϕ which achieves this winning result. [6]

END OF QUESTION PAPER