NZEST
NEW ZEALAND EDUCATION & SCHOLARSHIP TRUST
ACADEMIC ACHIEVEMENT BY CHOICE

NZEST SCHOLARSHIP EXAMINATION 1995

MATHEMATICS WITH CALCULUS

Time allowed: THREE hours

Ten minutes extra are allowed for reading this paper

Notes for Candidates

1 This examination consists of SIX questions, each worth 20 marks

2 Please attempt ALL SIX questions

3 You will get credit for every question (or part question) you do

4 Full marks can be obtained from five questions

5 Please show all working

6 One sheet of graph paper is available for use if required - if you use the graph paper, please tie it into your answer book

7 Calculators may be used

8 Tables of formulae are located on pages 10-12: these are detachable

Question 1

For any angle θ the expression $cis\theta$ is an abbreviation for $\cos\theta + i\sin\theta$. In a similar manner we can define $sic\theta = \sin\theta + i\cos\theta$.

(a) (i) [2] Draw an Argand diagram and on it show the complex numbers

$$z_1 = cis\left(\frac{\pi}{6}\right) \quad \text{and} \quad z_2 = sic\left(\frac{\pi}{6}\right).$$

(ii) [1] What transformation of the plane will always map $sic\theta$ on to $cis\theta$?

(b) [3] Prove $sic\theta = i\, cis(-\theta)$.

(c) [3] Using your result from (b), prove that $sic^n\theta = i^n cis(-n\theta)$.

(d) (i) [2] The equation $z^5 = i$ has five solutions on the complex plane. Show $sic(0)$ is a solution. Hint : use (c).

In a similar manner it can be shown that the other solutions are given by

$$sic\left(\frac{2\pi}{5}\right),\quad sic\left(\frac{4\pi}{5}\right),\quad sic\left(\frac{6\pi}{5}\right) \quad \text{and} \quad sic\left(\frac{8\pi}{5}\right).$$

(ii) [2] Give working to show that $sic\left(\frac{2\pi}{5}\right)$ is a solution.

(e) [4] Solve the equation $z^5 = i$ in the usual way without mentioning $sic\theta$. Give your answers in the $cis\theta$ notation, where $0 \le \theta \le 2\pi$.

(f) [3] Sketch another Argand diagram, this one showing the 5th roots of i, (ie the solutions of $z^5 = i$). Beside each root on the diagram write its value in both the $cis\theta$ and the $sic\theta$ forms.

Question 2

The series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad \text{can be used to evaluate the number } e.$$

(a) [1] Let $x = 1$. Write out an infinite series expansion for e.

To evaluate e we do not have enough time to add up all the terms in this series. So the following two methods can be used instead.

Method 1 : Use the sum of the first n terms in the series only. That is

$$s(n) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}.$$

(b) [1] Estimate the value of e by evaluating $s(4)$.

(c) [2] Prove that if $s(n)$ is used to estimate e, the answer will always be too small.

Method 2 : This is similar to method 1, but the last term is repeated thus

$$s^*(n) = s(n) + \frac{1}{(n-1)!}.$$

(d) [1] Estimate the value of e using $s^*(4)$.

(e) (i) [3] Prove that $\dfrac{1}{n!} = \dfrac{1}{2n!} + \dfrac{1}{4n!} + \dfrac{1}{8n!} + \dfrac{1}{16n!} + \ldots$

(ii) [3] Hence, or otherwise, prove that

$$\frac{1}{(n-1)!} > \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots \qquad \text{where } n > 1 \text{, is an integer.}$$

(f) [3] Using the result from (e) (ii), prove that if $s^*(n)$ is used to estimate e, then the answer will always be too large when $n > 1$.

Since $s(n)$ is always a little smaller than e, and $s^*(n)$ is always a little larger than e, this means that the interval $(s(n), s^*(n))$ must contain the number e for all positive integers n greater than 1, no matter how large.

(g) [1] Write down the interval $(s(n), s^*(n))$ for $n = 4$.

(h) [2] Find the length of the interval $(s(n), s^*(n))$ and show this tends to 0 as $n \to \infty$.

(i) [3] A fraction is defined as a number of the form $\dfrac{a}{n}$, where a and n are integers.

Show that any fraction can be written in the form $\dfrac{A}{n!}$, where A and n are integers. Show that both $s(n)$ and $s^*(n)$ can be written in this form. Explain briefly how this implies the number e cannot be a fraction.

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Question 3

A mathematician is writing a computer program with the language Hypercard 2.0 on an Apple computer. Hypercard 2.0 produces its output on electronic images called *cards*. A card is a 640x480 rectangle on the computer's screen with measurement in units called *pixels*. One pixel is approximately 0.4 mm. The origin (0,0) is at the top left hand corner of the screen. Refer figure 1.

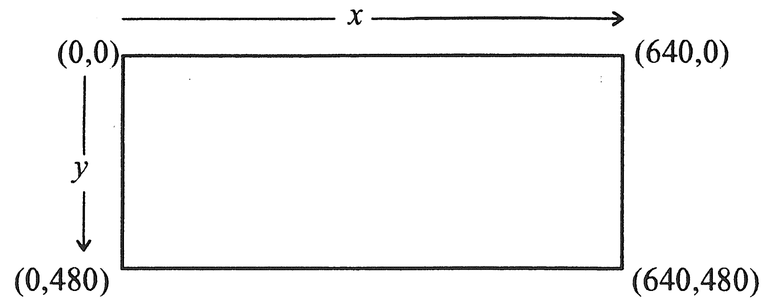

Figure 1 : The general appearance of a card with the (x,y) coordinate system.

(a) When the computer draws a graph on a card only those portions whose coordinates lie within the boundaries of the card are drawn. Make two copies of a card and sketch the graphs of :

(i) [1] $y = 2x + 100$

(ii) [1] $x^2 + y^2 = 320^2$.

(b) The mathematician decides to use part of the card to show text. The card is divided by a line connecting the points (320,0) and (320,480). The part of the card to the left of this line will be used for text. Graphs will now only be shown to the right of this line. A new coordinate system will be used for the graphical portion of the card. The new origin (0,0) is to be at the centre of the rectangle to the right of the dividing line. The axes in this new coordinate system are called the *X*- and *Y*- axes. They are drawn so as to each have maximum and minimum values of 10 and -10. Refer figure 2.

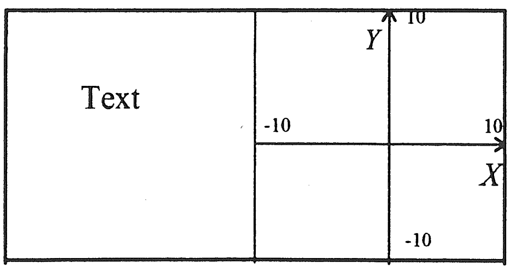

Figure 2 : A card divided into text and graph parts with the (X,Y) coordinate system.

Notice that although the computer user will see the graph on the (X,Y) axes, the instructions written in the computer program must still be written in the original (x,y) coordinate system.

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(i) [1] What is the equation of the dividing line according to the (x,y) system for the whole card (the original system of coordinates as in (a)) ?

(ii) [1] What is the equation of the dividing line according to the new (X,Y) system for the graphics portion of the card ?

(iii) [2] Give the equation of the X-axis first in the (x,y) system, and then in the (X,Y) system.

(c) Each point in the graphics part of the card now has two possible sets of coordinates, which we will refer to as (x,y) and (X,Y).

(i) [2] Obtain an equation for X in terms of x.

(ii) [2] Similarly find an equation for Y in terms of y.

(iii) [2] Prove that there is no point on the graphics part of the screen that has the same coordinates in both systems of coordinates.

(d) The mathematician programs the computer to draw the line $Y = 2X + 1$.

(i) [1] Make a sketch of a card showing the graph as the computer would display it.

(ii) [2] What is the equation of this line using the (x,y) coordinate system ?

(e) The mathematician wants to draw the circle $X^2 + Y^2 = 25$,
but when it appears on the graphics part of the card it does not really look like a circle.

(i) [2] Draw a scale diagram of the card showing the graph as it will appear.

(ii) [1] What is the equation of this graph if the (x,y) coordinates were used ?

(iii) [2] If -10 and 10 remain the minimum and maximum values of the X-axis, what must the minimum and maximum values of the Y-axis be if the graph of $X^2 + Y^2 = 25$ is to be a true circle ?

Question 4

A bungy jumper jumps vertically off the middle of a high bridge over a river and is filmed by a spectator with a video camera. The spectator wants to work out the best position at which to stand so as to obtain the clearest possible film of the jump. The diagram below shows three important positions. They are the position S of the spectator, the beginning point B of the jump and the position J of the bungy jumper t seconds after the beginning of the jump.

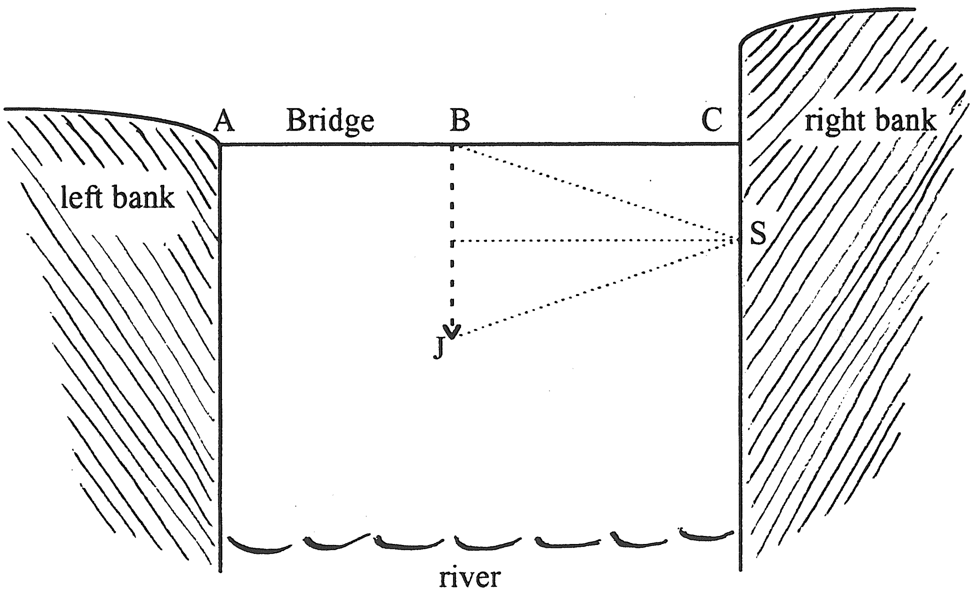

Figure 3 : The bungy jump showing the bridge AC, with the spectator at S.

(a) [3] Copy figure 3 marking clearly on it the following four important measures : the angle α between the horizontal and the line of sight from S to J (take α to be positive when J is below the level of S); the width w metres of the gap between S and the line of fall of the jumper; the difference c metres in height between S and B (take c to be positive when S is below the level of B); and the distance h metres that the jumper has fallen after t seconds. In what follows you may assume that

$h = \dfrac{gt^2}{2}$, where g is the acceleration due to gravity.

The fall is in two stages : (a) free fall during which $h = \dfrac{gt^2}{2}$, and (b) retarded fall after the elastic rope begins to slow the jump. In this question we are only interested in the free fall part of the jump, as the high speed attained here is the most likely cause of blurring on the video tape.

(b) [3] Use your diagram in (a) to explain why $\alpha = \tan^{-1}\dfrac{gt^2 - 2c}{2w}$.

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(c) [4] Use the formula $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

to show

$$\frac{d\alpha}{dt} = \frac{4wgt}{4w^2 + \left(gt^2 - 2c\right)^2}.$$

(d) [4] $\frac{d\alpha}{dt}$ is called the *angular velocity*. It measures the rate at which the angle α increases as the jumper falls. High values of angular velocity are likely to cause blurring of the video tape. As the jumper falls, the angular velocity increases for a while, reaches a maximum, and then decreases again. The maximum rate occurs, of course, when $\frac{d^2\alpha}{dt^2} = 0$. Solve the equation $\frac{d^2\alpha}{dt^2} = 0$ for t^2, and hence verify that the maximum rate is reached when

$$\alpha = \tan^{-1} \frac{\sqrt{3w^2 + 4c^2} - 2c}{3w}.$$

From practical consideration of the situation described, it may be assumed the stationary value obtained is in fact a maximum value. You are not required to perform a second derivative test (or equivalent).

(e) [2] Prove that if the spectator with the video camera is at the level of the bridge then the maximum value of the angular velocity occurs when $\alpha = \frac{\pi}{6}$, provided this angle is reached before the rope begins to slow the jumper down.

(f) [2] Find how far above or below (specify which) the level of the bridge the spectator should be located in order that the maximum angular velocity occurs when $\alpha = \frac{\pi}{4}$.

(g) [2] By referring to your mathematical calculations, explain why your answers to (e) and (f) would be unchanged if the bungy jump were performed somewhere at high altitude where the value of g, the acceleration due to gravity, is different.

Question 5

As a project for a science fair, a group of engineering students from Wellington decide to build a tower. The design they choose is obtained by taking the graph of $y = 1/x$ for $y \in [1, h]$, and rotating it about the y-axis. Units are metres.

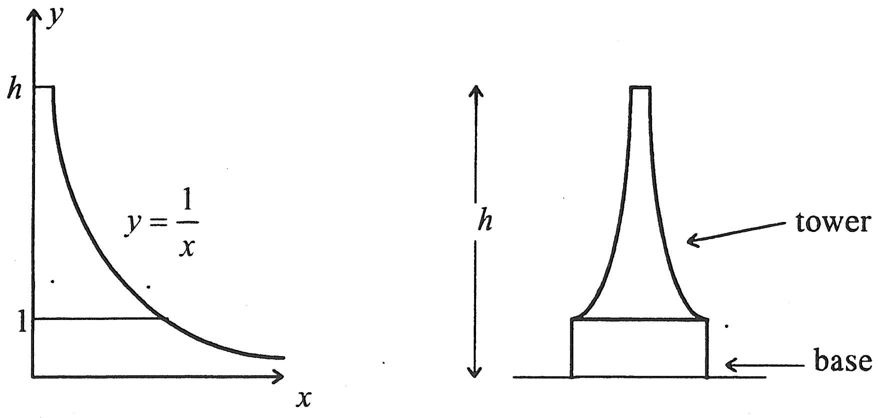

Figure 4 : (a) The graph (b) The tower

The tower will sit on a concrete base 1 metre high. So the height of the top of the tower above ground level will be h metres.

(a) [4] Using the method of volumes of revolution, find the volume of the tower as a function of h. Do not include the volume of the base.

The students decide to construct the tower out of solid carbon fibre, of which they have 3.2 m^3 available. Since they know the Auckland casino has a tower of 328 m, they decide that h should be 329m.

(b) [2] Show that they have enough carbon fibre for a tower of this height.

(c) [2] Show that they have enough carbon fibre to construct a tower for any value of h, no matter how large.

After the students have decided that h is to be 329m, the city council objects that such a tower would be a hazard to low-flying aircraft. They insist that it must be painted with a 1mm thick coat of bright orange paint.

(d) [5] What volume of paint is needed ? [Hint: use volumes of revolution again. It is adequate to measure the thickness of the paint in a horizontal dimension.]

One way to estimate the area of a surface is to divide the total amount of paint used by the thickness of the paint.

(e) [2] Use this method, along with your answer to (d) to estimate the surface area of the tower.

(f) [5] When completed there will be two main dangers to the tower. It may collapse under its own weight, or it may get blown down. By referring to your answers to (b) and (e) above, decide which of these is likely to be the greater danger. Would your answer remain valid if the value of h were increased even further ? [Hint : Weight is proportional to volume, while wind pressure is proportional to surface area.]

Question 6

A biologist needs a steady supply of yeast to support some experiments. She begins with B yeast cells. The growth of the yeast cells is controlled by the differential equation

$$\frac{dN}{dt} = kN(4B - N)$$

where k is a constant and N is the number of yeast cells t hours after the beginning.

(a) [3] Show that $N = \dfrac{4Be^{4kBt}}{3 + e^{4kBt}}$ is the particular solution to the differential equation

$$\frac{dN}{dt} = kN(4B - N) \quad \text{which satisfies the condition that } N = B \text{ when } t = 0.$$

(b) [2] Using the particular solution in (a), find the value of t for which $N = 2B$.

(c) [2] Show that if $t = \dfrac{1}{4kB}\ln\left(\dfrac{3}{2}\right)$, then $N = \dfrac{4B}{3}$.

In a similar way it can be shown that if $t = \dfrac{1}{4kB}\ln(9)$, then $N = 3B$. You do not have to prove this. Three different harvesting procedures are now considered.

(d) [3] First suppose that the biologist allows the number of yeast cells to grow to $3B$, and then harvests $2B$ of the cells, and this process is constantly repeated. Determine the average number of cells harvested per hour.

(e) [3] The second procedure is similar to that in (d), but now harvesting occurs when $N = 2B$, and only $\dfrac{2B}{3}$ cells are harvested each time. Find the average number of cells harvested per hour.

(f) [3] In the third procedure the cells are continuously harvested at a constant rate which is selected so that the number of cells, N, is held constant at $2B$. Find the number of cells harvested per hour in this case too.

(g) [1] Which of these three harvesting techniques gives the greatest harvest of cells ?

(h) [3] Using the second derivative test, or other suitable test, find the value of N at which the rate of growth of the yeast cells is a maximum. Use this information to prove that your choice in (g) is not merely the best of the 3 methods considered, but is in fact the best of all possible methods.

END OF QUESTION PAPER