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NZEST SCHOLARSHIP EXAMINATION 1996

MATHEMATICS WITH CALCULUS

Tuesday 12 November 1996, 2pm

To issue: Standard NZEST Answer Book

Time allowed: THREE hours

Notes for Candidates

1. This examination consists of **SIX** questions, each worth 20 marks
2. Please attempt **ALL SIX** questions
3. You will get credit for every question (or part question) you do
4. Full marks can be obtained from five questions
5. <u>Please show all working</u>
6. One sheet of graph paper is located on page 11: this sheet is detachable. If you use it, please ensure you tie it into your answer book.
7. Calculators may be used, except as stated in question 3
8. Tables of formulae are supplied as a centrefold sheet: this sheet is detachable.

Question 1

In this question n is a positive integer, with $n > 1$.

(a) Suppose $\omega = \text{cis}\theta$ is an n-th root of 1, that is $\omega^n = 1$.
Prove that $\overline{\omega} = \text{cis}(-\theta)$, the complex conjugate of ω, is also an n-th root of 1. [1]

(b) Draw an Argand diagram to show all possible values of the complex number ω, for the case when $\omega^5 = 1$. [2]

(c) Draw another Argand diagram to show all possible values of the complex number ω, for the case when $\omega^{10} = 1$. [1]

(d) Prove that if n is even and $\omega^n = 1$, then $-\omega$ is also an n-th root of 1. [2]

(e) Hence show that if n is even then the sum of all the n-th roots of 1 is 0. [3]

(f) (i) Let $f(x) = 1 - x^n$. Evaluate $f(1)$, and explain why this tells us that $(1 - x)$ is a factor of $f(x)$. [2]

(ii) Verify the factorisation $f(x) = (1-x)(1 + x + x^2 + \dots + x^{n-1})$ [1]

(g) (i) Let $z = \text{cis}\left(\frac{2\pi}{n}\right)$. Evaluate $f(z)$, where f is the same function as in part (f). [1]

(ii)Show why the result in part (f) implies that $1 + z + z^2 + \dots + z^{n-1} = 0$ [1]

(h) Use the result from (g) to prove that the sum of all the n-th roots of 1 is 0 for all positive integers n greater than 1 (even or odd). [2]

(i) Hence prove that

$$cis\left(\frac{\pi}{5}\right) + cis\left(\frac{3\pi}{5}\right) + cis\left(\frac{5\pi}{5}\right) + cis\left(\frac{7\pi}{5}\right) + cis\left(\frac{9\pi}{5}\right) = 0.$$ [2]

(j) Generalise the results you have obtained above to prove that the sum of the n-th roots of any complex number is always 0, for $n > 1$. [2]

Question 2

(a) Sketch three graphs of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for the cases :

(i) $a > b$
(ii) $a = b$
(iii) $a < b$. [3]

(b) Sketch a graph to show the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a > b$, and the circle lying within this ellipse which *touches* the ellipse at the two points where the line $x = c$ intersects the ellipse, where $0 \le c < a$. [1]

(c) Explain why the circle in (b) has an equation of the form $(x\text{-}d)^2 + y^2 = R^2$, where d and R will be suitable non-negative numbers. [1]

(d) (i) By implicit differentiation of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, find the gradient of each tangent to the ellipse in (b) at the points where $x = c$. [2]

(ii) Similarly, by implicit differentiation of the equation $(x\text{-}\ d)^2 + y^2 = R^2$, find the gradient of each tangent to the circle in (b) at the same points. [2]

(iii) Form an equation by combining your answers to (i) and (ii). Hence obtain an expression for d in terms of a, b and c. [3]

(iv) Similarly, obtain an expression for R in terms of a, b and c. [2]

(e) (i) Show that if $c = 0$, then the radius of the circle will be b. [1]

(ii) Show that in the limit when $c \to a$, the circle's radius approaches $\frac{b^2}{a}$. [2]

(f) Let k be the ratio of the area of the circle to the area of the ellipse, that is k = [area of circle]/[area of ellipse]. Prove $\frac{b^3}{a^3} \le k \le \frac{b}{a}$.

[Hint : The area of the ellipse is given by the formula $A = \pi ab$.] [3]

Question 3

In the olden days, before the invention of electronic calculators, most people had great difficulty getting the correct answers to large numerical calculations. In 1614 the eccentric Scottish nobleman, John Napier, published the first book of logarithmic tables. He invented the word "*logarithm*". At first he used logarithms to base e, where e = 2.71828(6s.f.). These are called *natural* logarithms. Tables of natural logarithms list values of the function $f(x) = \log_e(x)$. A copy of such a table is given on page 5. The purpose of these tables was to simplify numerical calculations involving multiplication, division and obtaining powers or roots of large real numbers. *You must answer parts (a), (b) and (c) of this question without using your calculator.*

(a) A user of these tables would have found that $\log_e(1.3) = 0.26236$, and $\log_e(13) = 2.56495$. Explain how the user would have obtained these values from the given table on page 5. [Hint : notice the helpful information at the foot of the table] [2]

(b) (i) Using the table given, show that $\log_e(1.3 \times 1.2) = \log_e(1.3) + \log_e(1.2)$
[There may be a slight rounding error. Do not worry about this.] [1]

(ii) Calculate $2.43 \div 2.13$ to 3 significant figures using the table. Explain each step. [2]

After these natural logarithm tables appeared [Napier's very first table listed natural logarithms of the sine function for navigators and astronomers], some users complained that the use of the number e as the base was too confusing. So Napier and his friend Henry Briggs decided to construct a new set of logarithm tables to base 10. These were called *common* logarithms. It was found that the new tables could be easily found from the natural logarithm tables by using the formula $\log_a x = \dfrac{\log_b x}{\log_b a}$.

(c) Use this method to find the value of $\log_{10}(1.6)$ to 3 s.f.. Explain your working. [*Do not use a calculator*. You may use $\log_e(10) = 2.30$ (3s.f.) to save time.] [2]

(d) (i) Give one reason why mathematicians preferred natural logarithm tables. [1]

(ii) Give one reason why other people preferred common logarithm tables. [1]

(e) (i) Show how A^x can be written in the form e^{kx}. Hence show that if $f(x) = A^x$, then $f'(x) = \log_e(A)\, A^x$. [2]

(ii) Prove that the angle between the tangents to the curves $y = e^x$ and $y = 10^x$ at (0,1) is given by $\tan^{-1}\left(\dfrac{\log_e 10 - 1}{\log_e 10 + 1}\right)$. [Hint : Use the formula $\tan\theta = \pm\dfrac{m_1 - m_2}{1 + m_1 m_2}$, where m_1 and m_2 are the slopes of the two lines intersecting at angle θ.] [4]

THIS QUESTION IS CONTINUED AT THE TOP OF PAGE 5

(f) Show that the area between the curves $y = e^x$ and $y = 10^x$ for $-k \le x \le 0$ is not greater than $\dfrac{\log_e 10 - 1}{\log_e 10}$, no matter how large k is. [5]

Table of Natural Logarithms

x	0	1	2	3	4	5	6	7	8	9
1.0	.00000	.00995	.01980	.02956	.03922	.04879	.05827	.06766	.07696	.08618
1.1	.09531	.10436	.11333	.12222	.13103	.13976	.14842	.15700	.16551	.17395
1.2	.18232	.19062	.19885	.20701	.21511	.22314	.23111	.23902	.24686	.25464
1.3	.26236	.27003	.27763	.28518	.29267	.30010	.30748	.31481	.32208	.32930
1.4	.33647	.34359	.35066	.35767	.36464	.37156	.37844	.38526	.39204	.39878
1.5	.40547	.41211	.41871	.42527	.43178	.43825	.44469	.45108	.45742	.46373
1.6	.47000	.47623	.48243	.48858	.49470	.50078	.50682	.51282	.51879	.52473
1.7	.53063	.53649	.54232	.54812	.55389	.55962	.56531	.57098	.57661	.58222
1.8	.58779	.59333	.59884	.60432	.60977	.61519	.62058	.62594	.63127	.63658
1.9	.64185	.64710	.65233	.65752	.66269	.66783	.67294	.67803	.68310	.68813
2.0	.69315	.69813	.70310	.70804	.71295	.71784	.72271	.72755	.73237	.73716
2.1	.74194	.74669	.75142	.75612	.76081	.76547	.77011	.77473	.77932	.78390
2.2	.78846	.79299	.79751	.80200	.80648	.81093	.81536	.81978	.82418	.82855
2.3	.83291	.83725	.84157	.84587	.85015	.85442	.85866	.86289	.86710	.87129
2.4	.87547	.87963	.88377	.88789	.89200	.89609	.90016	.90422	.90826	.91228
2.5	.91629	.92028	.92426	.92822	.93216	.93609	.94001	.94391	.94779	.95166
2.6	.95551	.95935	.96317	.96698	.97078	.97456	.97833	.98208	.98582	.98954
2.7	.99325	.99695	1.00063	1.00430	1.00796	1.01160	1.01523	1.01885	1.02245	1.02604
2.8	1.02962	1.03318	1.03674	1.04028	1.04380	1.04732	1.05082	1.05431	1.05779	1.06126
2.9	1.06471	1.06815	1.07158	1.07500	1.07841	1.08181	1.08519	1.08856	1.09192	1.09527
3.0	1.09861	1.10194	1.10526	1.10856	1.11186	1.11514	1.11841	1.12168	1.12493	1.12817
3.1	1.13140	1.13462	1.13783	1.14103	1.14422	1.14740	1.15057	1.15373	1.15688	1.16002
3.2	1.16315	1.16627	1.16938	1.17248	1.17557	1.17865	1.18173	1.18479	1.18784	1.19089
3.3	1.19392	1.19695	1.19996	1.20297	1.20597	1.20896	1.21194	1.21491	1.21788	1.22083
3.4	1.22378	1.22671	1.22964	1.23256	1.23547	1.23837	1.24127	1.24415	1.24703	1.24990
3.5	1.25276	1.25562	1.25846	1.26130	1.26413	1.26695	1.26976	1.27257	1.27536	1.27815
3.6	1.28093	1.28371	1.28647	1.28923	1.29198	1.29473	1.29746	1.30019	1.30291	1.30563
3.7	1.30833	1.31103	1.31372	1.31641	1.31909	1.32176	1.32442	1.32708	1.32972	1.33237
3.8	1.33500	1.33763	1.34025	1.34286	1.34547	1.34807	1.35067	1.35325	1.35584	1.35841
3.9	1.36098	1.36354	1.36609	1.36864	1.37118	1.37372	1.37624	1.37877	1.38128	1.38379
4.0	1.38629	1.38879	1.39128	1.39377	1.39624	1.39872	1.40118	1.40364	1.40610	1.40854
4.1	1.41099	1.41342	1.41585	1.41828	1.42070	1.42311	1.42552	1.42792	1.43031	1.43270
4.2	1.43508	1.43746	1.43984	1.44220	1.44456	1.44692	1.44927	1.45161	1.45395	1.45629
4.3	1.45862	1.46094	1.46326	1.46557	1.46787	1.47018	1.47247	1.47476	1.47705	1.47933
4.4	1.48160	1.48387	1.48614	1.48840	1.49065	1.49290	1.49515	1.49739	1.49962	1.50185
4.5	1.50408	1.50630	1.50851	1.51072	1.51293	1.51513	1.51732	1.51951	1.52170	1.52388
4.6	1.52606	1.52823	1.53039	1.53256	1.53471	1.53687	1.53902	1.54116	1.54330	1.54543
4.7	1.54756	1.54969	1.55181	1.55393	1.55604	1.55814	1.56025	1.56235	1.56444	1.56653
4.8	1.56862	1.57070	1.57277	1.57485	1.57691	1.57898	1.58104	1.58309	1.58515	1.58719
4.9	1.58924	1.59127	1.59331	1.59534	1.59737	1.59939	1.60141	1.60342	1.60543	1.60744

Note : $\log_e 10 = 2.30259$

Question 4

Today, 12 November 1996, the trees of a certain native forest in New Zealand carry 1000 million leaves. New leaves are being produced at the constant rate of 3 million leaves per day. Unfortunately, this forest is also home to a population of hungry possums. At present there are 1000 possums living in the forest, and they are eating and breeding energetically. Let P be the number of possums [in thousands]. The value of P will grow exponentially according to the differential equation

$$\frac{dP}{dt} = \frac{P}{200},$$

where t is time measured in days starting from today, and 200 is a factor expressing the rate at which these possums are breeding. Each possum eats approximately 1000 leaves per day. Let L be the number of leaves in the forest [in millions]. The value of L will be controlled by the differential equation

$$\frac{dL}{dt} = 3 - P .$$

(a) Solve the differential equation $\frac{dP}{dt} = \frac{P}{200}$, and obtain the particular solution given $P = 1$ when $t = 0$. Express your answer in the form of a formula which will determine the number of possums in the forest t days after today. [4]

(b) Estimate the number of possums there will be on 20/2/97, ie when $t = 100$. [2]

(c) By referring to the differential equation $\frac{dL}{dt} = 3 - P$, or otherwise, explain why the number of leaves in the forest will increase as long as P is less than 3, but when P becomes greater than 3 the number of leaves will decrease. Use this information to find the number of days from today until the time when the number of leaves on the trees reaches its maximum. [3]

(d) (i) Verify that $L = 3t - 200e^{\frac{t}{200}} + C$ is a general solution of the differential equation $\frac{dL}{dt} = 3 - P$. [2]

(ii) Evaluate the constant C by reference to the initial conditions given in the information above. [2]

(e) What will be the maximum number of leaves in this forest ? [3]

(f) The forest will collapse and die if the number of leaves drops below 300 million. Show that this collapse will take place during the month of March, 1998, unless drastic measures are taken to control the size of the possum population. [4]

CONTINUED

Question 5

(a) Using the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, find the derivative of the function $f(x) = (2x)^2$ by the first principles method. [3]

(b) (i) Show that $(u-v)(u^2 + uv + v^2) = u^3 - v^3$. [1]

(ii) Find the derivative of the function $g(x) = \sqrt[3]{x}$, when $x > 0$, by first principles. [Hint : Let $u = \sqrt[3]{x+h}$, and $v = \sqrt[3]{x}$] [3]

(iii) Prove $g(x)$ is not differentiable when $x = 0$. [Hint : One way to do this is to attempt to find the derivative of $g(x)$ using first principles when $x = 0$.] [3]

(c) A piece-wise function, $h(x)$, is defined on the real numbers in the following way :

$$h(x) = \begin{cases} k - (x+m)^2, \ldots\ldots x < -1 \\ 3x^{\frac{2}{3}}, \ldots\ldots\ldots\ldots -1 \le x \le 1, \\ k - (x-m)^2, \ldots\ldots\ldots x > 1 . \end{cases}$$

(i) Find values for k and m which will cause the function h to be continuous at every point of its domain and also to be smooth (ie differentiable) at every real number except 0. [4]

(ii) Find all x-intercepts of the function $h(x)$. [2]

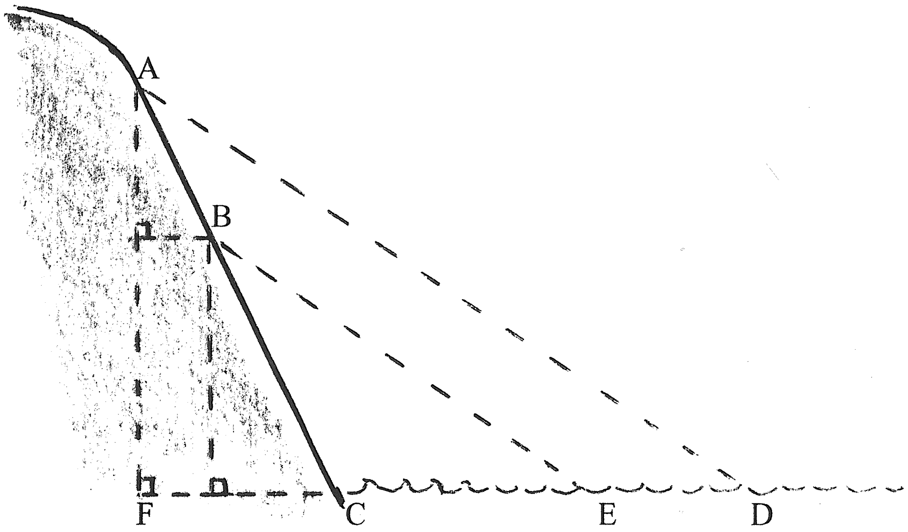
(iii) Find the area bounded by the x-axis and that part of the graph of $y = h(x)$ which is above the x -axis. [4]

[Note : The first principles method is not required for part (c).]

PLEASE TURN OVER

Question 6

Pat, a scientist involved in a research project to study ocean waves, has installed two cameras at points A and B high on a cliff-face which slopes at an angle of $\pi/3$ to the horizontal. Point A is directly up the cliff from B, and the distance AB is 60 metres. C is the point at the shore line directly down from A and B. At first each of the cameras is fixed at an angle of $\pi/4$ to the horizontal. The camera at A is focused on a point D on the surface of the sea directly off-shore from C. Similarly the camera at B is focused on a point E on the surface of the sea with the points ABCED coplanar. Let H be the height of A above sea level, and h be the height of B above sea level. Let W be the distance a wave will travel from point D to the shore at C, and let w be the similar distance from E to the shore. The following diagram [not to scale] summarises the positions of the important points in this problem.



(a) Copy the diagram. Mark on it the lengths H, h, W and w. Mark two angles which are $\pi/3$, and another two which are $\pi/4$. Also indicate which length is 60 metres. [2]

(b) (i) Form an equation relating H to W. [Hint : Find FC and FD in terms of H.] [2]

(ii) Find a similar equation relating h to w. [1]

After some time Pat observes that the waves travel at a constant velocity. A wave recorded at D reaches the shore at C 15 seconds later, while a wave seen at E takes only 12 seconds to reach the shore.

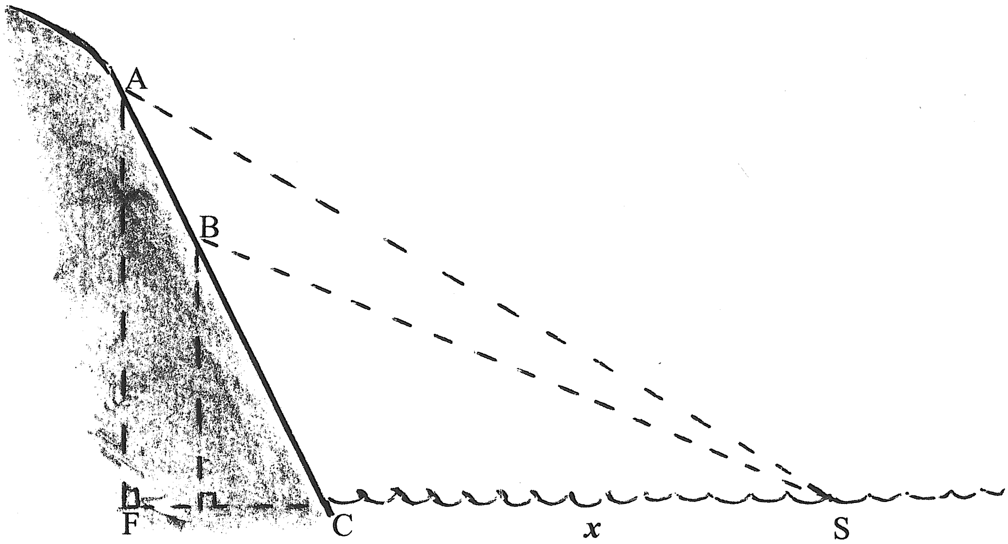
(c) Explain why Pat can now conclude that $W/w = 15/12$. Hence, using your results from (b), find an equation relating H to h. [3]

(d) By applying trigonometry to a suitable right-angled triangle, obtain a different equation relating H to h. Hence determine H. [3]

(e) How fast are the waves travelling ? [2]

THIS QUESTION IS CONTINUED ON PAGE 9

After several months Pat runs out of research funds and decides to make some money by using the equipment to film chewing-gum advertisements for television. A surfer rides a surf board on the face of a wave directly towards the shore. Both cameras remain focused on the surfer throughout the ride, and so have to be rotated carefully. Let α be the angle the camera at A is inclined to the horizontal, and let β be the similar angle for the camera at B. Refer to the following diagram where point S represents the location of the surfer.



(f) Using the variable x, where $x = \text{CS}$, show that $\dfrac{d\alpha}{dt}$, the rate of rotation of the camera at A, is given by

$$\frac{d\alpha}{dt} = \frac{-3H}{3H^2 + \left(H + \sqrt{3}x\right)^2} \cdot \frac{dx}{dt}. \qquad [2]$$

(g) Write down a similar expression for $\dfrac{d\beta}{dt}$, again using the variable x. [1]

(h) Hence or otherwise, prove that angle ASB has its maximum value when the surfer is $120\sqrt{5}$ metres from the shore. [Hint : Try to form a suitable equation in the variable x, and then solve it. The second derivative test is not required.] [4]

END OF QUESTION PAPER