NZEST

NEW ZEALAND EDUCATION
& SCHOLARSHIP TRUST

ACADEMIC ACHIEVEMENT BY CHOICE

NZEST SCHOLARSHIP EXAMINATION 1997

MATHEMATICS WITH CALCULUS

TUESDAY 11 NOVEMBER 1997, 2PM

TO ISSUE: STANDARD NZEST ANSWER BOOK

TIME ALLOWED: THREE HOURS

INSTRUCTIONS TO CANDIDATES

1 This examination consists of **SIX** questions, each worth 20 marks
2 Please attempt **ALL SIX** questions
3 You will get credit for every question (or part question) you do
4 Full marks can be obtained from five questions
5 Please show all working
6 One sheet of graph paper is located at the end of this booklet: this sheet is detachable. If you use it, please ensure you tie it into your answer book
7 Calculators may be used
8 Tables of formulae are supplied as a detachable centrefold sheet

Question 1

In this question we refer to a function f defined by $f(x) = (x^2 - 4x + 3)^2$.

(a) Evaluate $f(4)$. [1]

(b) In the method of first principles, $f'(x)$ is found by evaluating the limit :

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right].$$

Calculate the value of $f'(4)$ using this method. [4]

(c) $f'(x)$ can also be found without using first principles. Find $f'(x)$ for a general value of x by differentiating $f(x)$. You may find the chain rule useful. Once you have found an expression for $f'(x)$, use this to verify your answer to (b). [3]

(d) Find the equation of the tangent line to the curve $y = f(x)$ at the point $(4, f(4))$. [2]

(e) Find the equation of the normal line to the curve $y = f(x)$ at the point $(4, f(4))$. The normal line is the line at right angles to the tangent line. [3]

(f) Using the expression you found for $f'(x)$ in (c), show that $f'(1) = f'(3) = 0$. Briefly explain why you would expect this to be the case given that $x^2 - 4x + 3$ factors into the product $(x - 1)(x - 3)$. [3]

(g) Find the equation of a circle which has centre with both coordinates integers, which passes through the point $(4, f(4))$, and which also has as tangent at that point the line you found in (d). [4]

Question 2

In this question n is a positive integer and a is a positive real number. Both are constants.

(a) Use integration by substitution to evaluate the indefinite integral

$$\int x^{n-1} e^{-x^n}\, dx. \qquad [4]$$

(b) Evaluate the definite integral $\displaystyle\int_0^2 x\, e^{-x^2}\, dx$ in the following three ways, giving your answers to 3 decimal places :

(i) directly by substitution into the answer you obtained for part (a), [2]

(ii) approximately using the trapezium rule with 4 sub-intervals, and [3]

(iii) approximately using Simpson's rule, also with 4 sub-intervals. [3]

In (ii) and (iii) you should use 0.5, 1, and 1.5 as intermediate points.

(c) Sketch the region bounded by the curve $y = e^{-x^2}$ and the lines $x = -a$, $x = a$ and $y = 0$. [2]

(d) Write down and evaluate an expression involving integration which gives the volume of the solid obtained by revolving about the y-axis the region bounded by the curve $y = e^{-x^2}$ and the lines $x = 0$, $x = a$ and $y = 0$. [3]

(e) Explain why your answer to (d) would be unchanged if the region to be rotated were altered to the region bounded by the curve $y = e^{-x^2}$ and the lines $x = -a$, $x = a$ and $y = 0$. [2]

(f) By letting $a \to \infty$ in your answer to (d), show that the volume of the solid obtained by revolving about the y-axis the region bounded by the curve $y = e^{-x^2}$ and the x-axis and the y-axis is π. [1]

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Question 3

A new comet has been sighted and astronomers have calculated that its orbit is given by the equation

$$x^2 + 50y^2 = 50.$$

The distance units used throughout this question are *astronomical units*, where 1 astronomical unit is equal to the distance from the earth to the sun. It is known that the orbit of a comet is always an ellipse, with the sun situated at one of the foci points.

(a) Sketch a large graph to show the orbit of the comet. [2]

(b) The sun is situated at the point S which has coordinates (7,0). The earth is at point E with coordinates (7, 1). Mark the positions of the sun and the earth on your diagram in (a). The point where the comet gets closest to the sun is P, which has coordinates ($\sqrt{50}$, 0). Mark this point too on your diagram. For simplicity we assume the sun and the earth are at fixed points relative to the orbit of the comet. [3]

(c) Prove that the earth is outside the ellipse representing the orbit of the comet. [4]

(d) A Russian space research team decides to send a rocket to intercept the comet at point P. Assuming this rocket travels in a straight line from E to P, prove that the Russian rocket will cross the orbit of the comet in 2 places. Find the coordinates of both these points. Hint : Find the y-coordinates of the intersection points first. [5]

(e) An American NASA team also decides to launch a rocket, but their plan is different. They want their rocket to fly alongside the comet for as long as possible, and take TV pictures for nature programmes. The best way to achieve this is to send the rocket along a line which makes a tangent to the ellipse. Let T be the point on the ellipse such that ET is a tangent to the ellipse. On your diagram show 2 possible tangents from E to the ellipse. Mark T at the point on the ellipse where the distance ET is the shorter of the 2 possibilities. Notice that we are again assuming that the American rocket also travels in a straight line.
Let the point T have coordinates (u, v). We are going to find these coordinates by the following steps.

(i) Prove that the gradient of the tangent at any point (u, v) is given by $\dfrac{-u}{50v}$. [2]

(ii) Find another expression for the gradient of ET using the variables u and v. [1]

(iii) Complete your calculation of the actual values of u and v by making use of (i) and (ii) and also the equation of the ellipse. [3]

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Question 4

It is claimed that the following inequality is true.

$$\text{If} \quad 0 < x < 1 \quad \text{then} \quad \pi \le \frac{\sin(\pi.x)}{x(1-x)} \le 4 \ .$$

(a) Test the truth of this inequality when $x = 1/3$ and when $x = 3/4$. [4]

(b) We will now attempt to prove the right hand part of the inequality.

(i) Let $f(x) = \dfrac{\sin(\pi.x)}{x(1-x)}$, $0 < x < 1$. Find $f'(x)$. [2]

(ii) Given that $f(x)$ has only one turning point, show this occurs when $x = 0.5$. [4]

(iii) Prove that this turning point is a maximum. Hence deduce that $\dfrac{\sin(\pi.x)}{x(1-x)} \le 4$ on the interval $0 < x < 1$. [3]

(c) The remaining part of this question concerns the left hand part of the inequality.

(i) Show $\lim\limits_{x \to 0}\left(\dfrac{\sin(\pi.x)}{x(1-x)}\right) = \pi$. [2]

Useful hints : $\lim\limits_{x \to 0}\left(\dfrac{\sin(\pi.x)}{\pi.x}\right) = 1$ and $\dfrac{\sin(\pi.x)}{x(1-x)} = \dfrac{\pi.\sin(\pi.x)}{\pi.x(1-x)}$.

(ii) Show $\lim\limits_{x \to 1}\left(\dfrac{\sin(\pi.x)}{x(1-x)}\right) = \pi$. Hint : $\dfrac{\sin(\pi.x)}{x(1-x)} = \dfrac{\sin(\pi.(1-x))}{x(1-x)}$ [2]

(iii) We now extend the definition of function $f(x)$ to include end points.

$$\text{Let} \quad f(x) = \begin{cases} \pi ....when..x = 0 \\ \dfrac{\sin(\pi.x)}{x(1-x)} ....when..0 < x < 1 \\ \pi ....when..x = 1 \end{cases}$$

Prove that the minimum value of f on the interval $[0, 1]$ is π. [2]

Useful hint : A continuous function which is differentiable at every point on an interval except the end points achieves its maximum and minimum values either at a turning point, or at one of the end points of the interval.

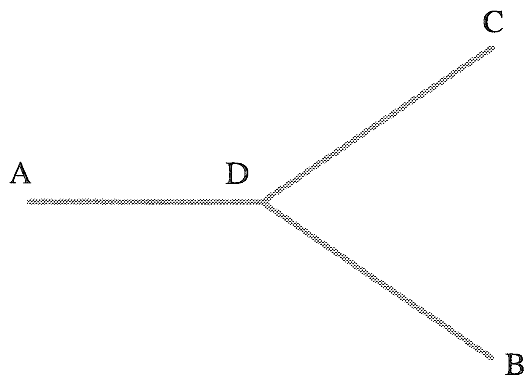
(iv) Using (iii) or otherwise, complete the proof of the original inequality :

$$\text{if} \quad 0 < x < 1 \quad \text{then} \quad \pi \le \frac{\sin(\pi.x)}{x(1-x)} \le 4 \ .$$ [1]

CONTINUED

Question 5

A road construction firm have a contract to build a motorway connecting three towns, A, B and C. As motorways are extremely expensive to build, they wish to design the plan so that the total length is as short as possible. The towns form an isosceles triangle, with AB = AC = 50 km, and BC = 60 km. As the problem is symmetric, they decide the optimal answer will be symmetric. They will build an intersection at a point D somewhere on the axis of symmetry, and will build three sections of motorway AD, BD and CD as shown in the diagram below.



(a) Use calculus to find the location of D which gives the shortest possible total length of motorway. Show that this shortest length is $40 + 30\sqrt{3}$ km. [8]

(b) Point D is called a *Steiner point* of triangle ABC. Most triangles have a Steiner point. Steiner points are very useful when dealing with shortest network problems such as (a) above. The angles formed at a Steiner point have a curious geometric property, which you may notice from your answer to (a).
Calculate the size of each angle formed at D by the line segments AD, BD and CD.

[2]

(c) Later the same company is given a contract to build another motorway linking 4 towns, E, F, G and H. Here EFGH forms a rectangle with EF = GH = 80 km, and FG = HE = 90 km. Make at least two diagrams showing possible ways such a motorway network could be designed. Calculate the total length of motorway construction required in each case. Find the plan which gives the shortest total length of motorway for all possible networks. Find this minimal distance. Give reasons to justify your answer.

[10]

Question 6

(a) The following function $f(z)$ is defined for any complex number z of the form $z = \cos\theta + i\sin\theta$, except 1. Remember that this definition of the domain of $f(z)$ applies throughout this question.

$$f(z) = 1 + \frac{4}{\bar{z} - 1}, \qquad z = \cos\theta + i\sin\theta, \qquad z \neq 1.$$

Here $\bar{z}$ denotes the conjugate of z. Thus $\bar{z} = \cos\theta - i\sin\theta$.

(i) Draw an Argand diagram to show the set of all complex numbers in the domain of the function $f(z)$. [2]

(ii) Given the identity $\dfrac{1}{(\cos\theta - i\sin\theta) - 1} = -\dfrac{1}{2}\left(1 - i\cot\dfrac{\theta}{2}\right)$, provided $\theta \neq 2n\pi$, where n is any integer, show that $f(z) = -1 + 2i\cot\dfrac{\theta}{2}$. [3]

(iii) Show that 1, z and $f(z)$ are represented by 3 collinear points on an Argand diagram for any complex number z in the domain of $f(z)$. [3]

(iv) Draw an Argand diagram. Let $z_1 = -i$. Mark the points representing 1, z_1 and $f(z_1)$ on your diagram. Hint : Use the results from (ii) and (iii) above. It is not necessary to calculate the value of $f(z_1)$. [2]

(b) Using De Moivre's theorem, find the 4th roots of unity by solving the equation $w^4 = 1$. Leave your answer in the form $\cos\theta + \mathrm{i}\sin\theta$. [3]

(c) Let w_1, w_2 and w_3 be the 4th roots of unity other than 1. Draw another Argand diagram showing all the 4th roots and $f(w_1)$, $f(w_2)$ and $f(w_3)$. [4]

(d) Let n be any integer greater than 2, and let $w_1, w_2 \ldots w_{n-1}$ be the nth roots of unity other than 1 itself. Show that $f(w_1), f(w_2) \ldots f(w_{n-1})$ are represented by n-1 points lying on a line segment of length $4\cot\dfrac{\pi}{n}$ in the Argand diagram. [3]

END OF QUESTION PAPER