Question 1.

(a) Suppose that $y = x(x+1)^2$.

(i) Find where $\frac{dy}{dx}$ is positive and where $\frac{dy}{dx}$ is negative. [3]

(ii) Find where the curve meets the x-axis. [2]

(iii) Draw x and y axes on a sheet of graph paper with $-2 \le x \le 1$ and $-2 \le y \le 4$. Then, using the information from (i) and (ii), sketch (do NOT plot) the graph of $y = x(x+1)^2$. Label the important points you found in (i) and (ii). [2]

(b) Consider the parametric equations $x = t^2$ and $y = t^3 + t$ as t ranges through the real numbers.

(i) Determine $\frac{dy}{dx}$ in terms of t. [2]

(ii) Hence show that the slope of the curve defined by these equations tends to $\pm\infty$ as t tends either to 0 or to $\pm\infty$. [2]

(c) (i) Show that elimination of the parameter t from the equations $x = t^2$ and $y = t^3 + t$ leads to the cartesian equation $y^2 = x(x+1)^2$. [2]

(ii) Using (a), (b) and (c)(i), or otherwise, and on the same sheet of graph paper as in (a)(iii), sketch the graph of the curve defined by $x = t^2$ and $y = t^3 + t$ for $-1 \le t \le 1$. Be sure to label the two graphs so that it is clear which answers (a)(iii) and which answers (c)(ii). [2]

(d) (i) Write down a formula which gives the distance from the point $(a, 0)$ $(a > 0)$ to the point $(t^2, t^3 + t)$. [2]

(ii) Show that if $a \le \frac{1}{2}$ then there is exactly one point on the curve defined by $x = t^2$ and $y = t^3 + t$ which is nearest to $(a, 0)$ but if $a > \frac{1}{2}$ then there are two such points. Your answer should include an explanation as to why these points are nearest to $(a, 0)$. [3]

Question 2.

John and Marsha are old friends. John is now postmaster on a space station orbiting a planet in a distant galaxy. Marsha is now the pilot of an IGNZP (Inter Galactic New Zealand Post) mail delivery starship.

The space station orbits the planet in a circle and the path of the starship lies in the same plane as this circle. In this question we are interested only in where the orbit of the space station and the path of the starship meet or cross. Taking a set of axes with origin the centre of the planet, the orbit of the space station can be represented by $x^2 + y^2 = R^2$ and the path of the starship by $y = k - x^2$ where $k > 0$.

(a) Sketch the graph of $x^2 + y^2 = 4$. [1]

(b) Sketch the graph of $y = 4 - x^2$. [1]

(c) (i) John and Marsha can fail to meet or they can meet at a number of points. What is the maximum number of points at which John and Marsha can meet? [1]

(ii) For each possibility from 0 up to the maximum possible, sketch a separate diagram showing how the orbit and path meet or cross. [3]

(d) In each case where there are exactly **three** points of contact

(i) Find the coordinates where John and Marsha can make contact in terms of R. [5]

(ii) Show that in this case $R > \frac{1}{2}$. [2]

(e) In the case where there are exactly **two** points of contact and the circle is entirely within the parabola let a point of contact be (x_1, y_1).

(i) Show that $y_1 = \frac{1}{2}$ no matter what the value of R is. [4]

(ii) Show that Marsha must have set her path so that $k = R^2 + \frac{1}{4}$. [3]

Question 3.

(a) Two students, John and Hemi, argue about the areas formed by bending the same piece of wire into various regular polygons. Hemi says that an equilateral triangle has a smaller area than a square which in turn has a smaller area than a regular pentagon. Determine if Hemi is correct. [12]

(b) Maria claims that any regular polygon has an area less than the area of a circle of the same perimeter. She bases her argument on the fact that $x < \tan x$ for $0 < x < \frac{\pi}{2}$.

By using this fact or any other method prove her hypothesis. [8]

Question 4.

(a) (i) Solve the equation $x + \sqrt{1 - x^2} = \sqrt{2}$. [3]

(ii) Hence or otherwise solve the equation $\sin x + \cos x = \sqrt{2}$ for $-\pi \le x \le \pi$. [2]

(b) Define the function $\sec x$ in terms of $\cos x$ being careful to state any values of x for which your definition is not valid. [1]

(c) Using the definition given in (b) prove that $\sec^2 x = 1 + \tan^2 x$. [2]

(d) Using (c) or otherwise prove that $\tan(\cos^{-1} x) = \dfrac{\sqrt{1 - x^2}}{x}$. [4]

(e) Find an expression for z in terms of x which does not have any trigonometric function in it if $\sin^{-1} x = \tan^{-1} z$. [Hint: one approach is to use $\tan(\sin^{-1} x)$.] [3]

(f) Note that it is unlikely that a solution to the following makes use of parts (a) to (e). A hillside is in the shape of a plane sloping down from west to east and making an angle of 10° with the horizontal. A straight road goes across and up the hillside making an angle of 6° with the horizontal on the hillside. Figure (i) shows a vertical west-east slice through the hillside. Figure (ii) shows the hillside from above and slightly to the east. Neither figure is drawn to scale.

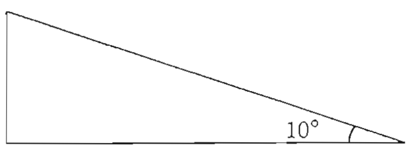


Figure (i)

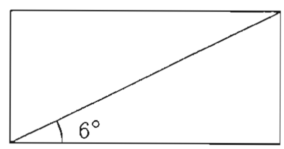


Figure (ii)

Copy Figure (iii) to whatever size is convenient. The plane shown is a horizontal plane. On it sketch how the line of the road lies on the hillside in 3 dimensions. Start by drawing the hillside using this plane as a base.

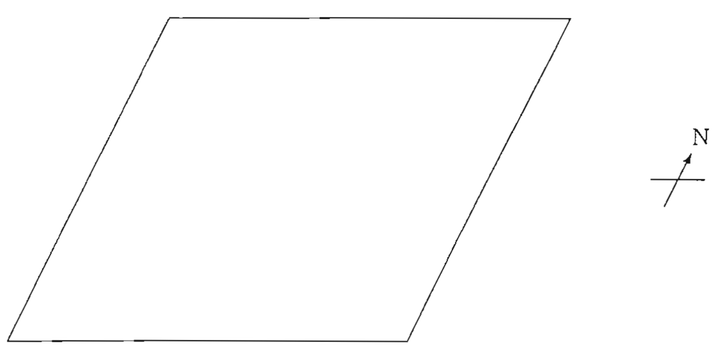
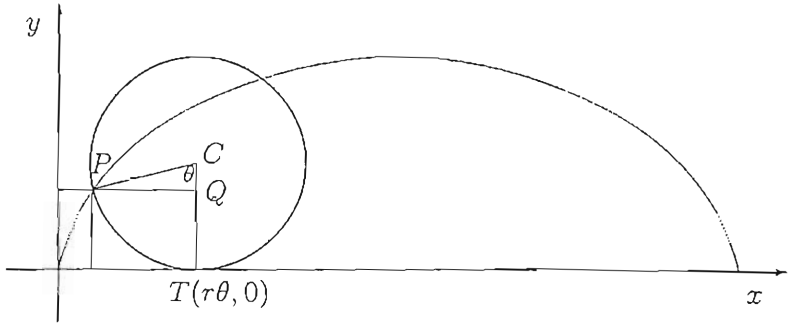


Figure (iii)

Find the angle of elevation of the road. [5]

Question 5.

(a) A wheel of radius r rolls along the x-axis beginning so that it is touching the x-axis at the origin. A point P is fixed on the circumference of the wheel and was at the origin. The curve traced out by P is called a cycloid. The diagram below shows the situation when the wheel has rotated through θ radians, and includes the first arch of the cycloid itself.



We confine our attention to the case where $0 \le \theta \le 2\pi$, in which case P traces out one arch of the cycloid as shown.

(i) Show that the coordinates of the point $P(x, y)$ are given by

$$x = r\theta - r\sin\theta, \qquad y = r - r\cos\theta.$$

[3]

(ii) Find $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ and hence show that $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}$. [3]

(iii) The area under the arch is $\displaystyle\int_0^{2\pi r} y\,dx$, which transforms to $\displaystyle\int_0^{2\pi} y\frac{dx}{d\theta}\,d\theta$ for the parameter θ. Calculate this area giving your answer in terms of π. [3]

(iv) Deduce from (iii) that the circle centred at $(\pi r, r)$ of radius r, which you may assume fits within the arch, divides the arch into 3 regions of equal area. [2]

(v) The length of the arch is $\displaystyle\int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$. Calculate this length showing all working. [3]

(b) By choosing suitable expressions for x and n in the expansion of $(1+x)^n$ prove the following results. It might help you to begin by writing out the expressions in full without the Σ sign.

(i) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. [1]

(ii) $\sum_{k=1}^{p} \binom{2p}{2k-1} = 1 + \sum_{k=1}^{p} \binom{2p}{2k}$. [2]

(iii) $\sum_{k=1}^{2m} (-1)^{k-1} \binom{4m}{2k-1} = 0$. [Hint: You may wish to use a complex number for x in this part.] [3]

Question 6.

The number π, the ratio of the circumference of a circle to its diameter, is of fundamental importance in the modern world. Over the past few millenia attempts have been made to express it as a rational number such as 3 or 22/7 and, more recently, to find rational numbers which are arbitrarily close to π. This question investigates the approximation of π by 22/7 and then goes on to look at one way of finding closer rational approximations.

Consider the rational function $\dfrac{x^4(1-x)^4}{x^2+1}$.

On expanding the numerator and denominator and then dividing we obtain:

$$\frac{x^4(1-x)^4}{x^2+1} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}.$$

(a) Evaluate $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx$. You may assume that $\displaystyle\int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4}$, which can be verified easily using the substitution $x = \tan\theta$ but you are not required to do this here. [3]

(b) Show that $0 < \displaystyle\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx < \frac{1}{2^8}$ as follows:

(i) Explain why $\dfrac{x^4(1-x)^4}{x^2+1} \geq 0$ for all x. [1]

(ii) Show that for all x strictly between 0 and 1, $\dfrac{x^4(1-x)^4}{x^2+1} > 0$. From this it follows that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx > 0$ but you are not asked to prove this. [1]

(iii) Explain why $\dfrac{x^4(1-x)^4}{x^2+1} < x^4(1-x)^4$ for all x strictly between 0 and 1. [2]

(iv) By finding the maximum of $x^4(1-x)^4$ on the interval from 0 to 1, show that

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx < \frac{1}{2^8}.$$

[3]

(c) (i) What does the result in (b) tell us about approximating π by 22/7? [2]

(ii) Explain the significance of the number 0 in its first appearance in

$$0 < \int_0^1 \frac{x^4(1-x)^4}{x^2+1}dx < \frac{1}{2^8}.$$

[1]