Question 1.

(a) (i) Sketch an equilateral triangle of side 2 units.

Hence, or otherwise, show that $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$. [2]

(ii) Sketch a right angled isosceles triangle whose equal sides enclose the right angle and are of length 1.

Hence, or otherwise, show that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. [1]

These numeric expressions are called **exact** values. Use exact values in the rest of this question.

(b) (i) Solve the equation $z^3 = 1$ over the set of complex numbers, giving the answers in polar form and expressing the angles as multiples of π. [3]

(ii) Draw these solutions on an Argand diagram. [1]

(iii) Use the solution to b(i) or use some other method to solve the equation $(z^2+1)^3 = 1$.
Give the answers in polar form and express the angles as multiples of π. [6]

(c) (i) Write $1+i$ in polar form. [1]

(ii) Prove $(1+i)^{4n}$ is real, n is assumed to be a whole number. [2]

(d) If n is a positive, whole number, the binomial expansion of $(a+b)^n$ is

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{k}a^{n-k}b^k + ... + b^n.$$

(i) Find the binomial expansion of $(1+i)^{4n}$. [2]

(ii) Use c(i), or otherwise, to show $\sum_{j=1}^{2n}\binom{4n}{2j-1}(-1)^j = 0$. [2]

Question 2.

(a) A comet approaches two planets in such a way that the difference between the comet's distances from the planets is always equal to d, a constant. Assume the planets can be taken as stationary over the time interval we are interested in.

Let the planets be at points $(-c,0)$ and $(c,0)$ in some cartesian coordinate system. Let the planet at $(c,0)$ be the planet the comet passes closest to. The comet passes between the two planets.

(i) Prove d is less then $2c$. [Hint: Let the comet's path cut the line joining the two planets at a distance of k from the origin.] [2]

(ii) Write an expression connecting the coordinates (x, y) of the comet with d, the difference between the distances of the comet from the planets. [2]

(iii) Obtain the equation of the comet's path, $\dfrac{4x^2}{d^2} - \dfrac{4y^2}{4c^2 - d^2} = 1$, from the expression in (ii).

[Hint: Rearrange the expression so that there is only one square root on the left hand side] [6]

(iv) State what sort of curve the comet is following. [1]

(b) This particular comet has locus $\dfrac{x^2}{25} - \dfrac{y^2}{200} = 1$.

(i) Find the value of d if the equation of the comet is to be written in the form of a(iii). [1]

(ii) Find the coordinates of the positions of the two planets. [1]

(c) A graph has parametric equations $x = 3 + 2\cos\theta$, $y = 1 + 3\sin\theta$.

(i) Eliminate θ and find the cartesian equations of the graph. [2]

(ii) Find the gradient of the tangent when $\theta = \dfrac{\pi}{4}$. [2]

(iii) Find the coördinates of the point where the normal to the curve when $\theta = \dfrac{\pi}{4}$, cuts the x-axis. [3]

Question 3.

(a) There is an discussion going on between Marsha and Hemi concerning functions which tend to a horizontal asymptote as $x \to \pm\infty$.

Hemi says that $f(x) = \dfrac{e^x + 1}{e^x - 1}$ has

(i) No turning points.
(ii) A vertical asymptote. [This is a value a such that $|f(x)| \to \infty$ as $x \to a$.]
(iii) A horizontal asymptote. [This is a value c such that $f(x) \to c$ as $x \to \pm\infty$.]
(iv) A derivative which tends to 0 as $x \to \pm\infty$.

Marsha says $f(x)$ does not have all these properties.

Determine if Hemi or Marsha is correct. [7]

(b) Hemi also believes all functions which have a horizontal asymptote have a derivative which tends to 0 as x tends to infinity. Marsha disagrees and says Hemi should have a look at the function defined by $g(x) = 2 - \frac{1}{x}\sin(x^2)$.

Determine if Hemi or Marsha is correct. [5]

(c) Sione, a friend of Hemi and Marsha, finds that $h(x) = \dfrac{x^2 + 1}{2x^2 + 3}$ is increasing for $x > 0$ and has a horizontal asymptote $y = c$.

(i) Show how Hemi might have proved that $h(x)$ is increasing for $x > 0$.
(ii) Find the value of c.
(iii) Find k such that $x > k$ implies that $h(x) > c - \dfrac{1}{100}$. [8]

Question 4.

(a) If v is any variable and Δv represents a small change in v, the percentage change in v is given by $\frac{\Delta v}{v} \times 100\%$.

Show that the small change corresponding to a 1% increase in v is given by $\Delta v = 0.01v$. [1]

(b) Let P represent the cost of a particular item of which D units are sold. P is called the price and D is called the demand. The elasticity of demand is defined by

$$E = \frac{P}{D}\frac{dD}{dP}.$$

(i) Using the fact that $\frac{\Delta D}{\Delta P} \approx \frac{dD}{dP}$, show that E is approximately equal to the percentage change in the demand D, due to a 1% increase in the price P. [3]

(ii) Explain why E is usually negative. [1]

For the remainder of this question, assume E is negative.

(iii) 1 Calculate E if price and demand are related by $D = 200 - 4P$ where $0 \le P \le 50$. [2]

2 Calculate the price for which $E = -1$. [1]

3 What is the significance of the restriction $0 \le P \le 50$? [1]

(iv) 1 The total revenue R is given by $R = PD$. Use implicit differentiation, or otherwise, to show that $\frac{dR}{dP} = D(E+1)$. [2]

2 Hence show that $|E| = 1$ corresponds to the maximum revenue. [2]
(You must justify why this is a maximum)

3 What is the corresponding relationship for $\frac{dR}{dD}$? [2]

(v) Consider the quadratic relationship connecting price and demand:
$P = a - bD^2$ where $a, b, P, D > 0$.

1 What additional condition needs to be stated? [2]

2 By calculating E, or otherwise, find the range of values of demand which correspond to an increasing revenue. [3]

Question 5.

(a) (i) 1 By means of a diagram, explain why $\sin^2 x + \cos^2 x = 1$. [2]

2 Hence prove that $1 + \tan^2 x = \sec^2 x$. [1]

(ii) Let $I_n = \int \tan^n x \, dx$.

1 By writing $\tan^n x = \tan^{n-2} x . \tan^2 x$, prove that

$$I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}.$$ [2]

2 Hence find an expression for $\int \tan^8 x \, dx$. [3]

(b) (i) Evaluate $\int \cos^5 x \, dx$ using the substitution $t = \sin x$. [3]

(ii) 1 If $f(x) = \cos^5 x$, what are $f(0)$, $f'(0)$ and $f''(0)$? [2]

2 Hence evaluate a, b and c in the approximation $\cos^5 x \approx a + bx + cx^2$.
Note, this is only valid for small values of x. [1]

3 Use this to give an approximation for $\int \cos^5 x \, dx$. [1]

(c) (i) Give the values for the gaps in the following table: [1]

x	$\cos^5 x$	$a + bx + cx^2$
0	1	. 1
0.1	0.975	0.975
0.2	0.904	0.9
0.3	0.796	0.775
0.4	0.663	0.6
0.5	0.521	0.375
0.6	0.383	0.1
0.7		-0.225
0.8	0.164	

(ii) Evaluate $\int_0^{0.6} \cos^5 x \, dx$ to three significant figures, using:

1 The exact integration in b(i). [1]

2 The expression in b(ii)3. [1]

3 Simpson's rule. [2]

Question 6.

(a) (i) Show that $\frac{d}{dt}\left(e^{\beta t}w\right) = e^{\beta t}\frac{dw}{dt} + e^{\beta t}\beta w$. [2]

(ii) Consider the differential equation $\frac{dw}{dt} = \alpha - \beta w$ where α, β are constants.

Show it has a solution $w = \frac{\alpha}{\beta} + ce^{-\beta t}$, where c is an arbitrary constant. Do this by multiplying through by $e^{\beta t}$ and then using the result in (i) above. [4]

(b) The Bertalanffy model for the growth of a fish is expressed in the form of the differential equation $\frac{dm}{dt} = Am^{2/3} - Bm$ where m is the mass of the fish at time t and A and B are constants. The first term on the right hand side of this equation represents the increase in mass of the fish due to nutrients and the second term represents the loss of mass due to respiration and is proportional to the mass of the fish.

(i) Reduce the differential equation $\frac{dm}{dt} = Am^{2/3} - Bm$ to one of the form $\frac{dw}{dt} = \alpha - \beta w$ by making the substitution $m = w^3$. [4]

(ii) Using the result you obtained in (a)(ii), obtain the general solution to the Bertalanaffy model in terms of m, A, B. [2]

(iii) Assume that the fish grows from zero mass. If the maximum mass a fish can attain is given by M, show the predicted growth in fish size is $m = M\left(1 - e^{-\beta t}\right)^3$ where $\beta = \frac{B}{3}$. [3]

(iv) Here is a sketch graph of this function.

1 Explain why it goes through the origin. [1]

2 Without using calculus, explain why the graph is increasing. [2]

3 Give the equation of the asymptote. [2]