Question 1 .

a Consider the function $f(x) = e^x$.

i Calculate the gaps left in this table to 4 dp.

x	f(x)
1	2.7183
1.25	3.4903
1.5	4.4817
1.75	
2	7.3891
2.25	9.4877
2.5	12.1825
2.75	15.6426
3	20.0855

x	f(x)
3.25	25.7903
3.5	
3.75	42.5211
4	54.5982
4.25	70.1054
4.5	90.0171
4.75	115.5843
5	

[1]

ii Evaluate $\int_1^5 f(x)dx$ to one decimal place.

[1]

iii Approximate $\int_1^5 f(x)dx$ to one decimal place, by using the Trapezium Rule and four sub intervals.

[2]

b Reasonable approximations to $\int_1^5 f(x)dx$ can be found as follows:

i Split the interval [1,5] into four sub intervals [1,2], [2,3], [3,4], [4,5].
Calculate $f(1.5)(2-1) + f(2.5)(3-2) + f(3.5)(4-3) + f(4.5)(5-4)$.

[2]

ii Find a better approximation by generalising the above to 8 sub intervals.

[3]

The method used in **2 b** above can be generalised as follows:

Let t_j be the mid-point of the sub interval $[x_j, x_{j+1}]$

and $\delta_j = \dfrac{b-a}{n}$ *be the width of* $[x_j, x_{j+1}]$

Then $\sum_{j=0}^{n-1} f(t_j)\,\delta_j \to \int_a^b f(x)\,dx$ *as* $n \to \infty$

This piece of information will be used to help solve the problem on the next page.

QUESTION 1 CONTINUES ON NEXT PAGE

c A new medical clinic has just opened. It is known that the fraction of patients who will still be receiving treatment from the clinic at the end of t months after their initial visit is given by the function $g(t) = e^{-t/20}$. The clinic at the time of opening accepts 300 people and plans to accept new patients at the rate of 10 per month. The following problem has to be solved: *How many patients will be receiving treatment from the clinic 15 months after it has opened ?* Answer the following questions in your attempt to solve this problem.

i How many of the original 300 people will still be receiving treatment in 15 months' time ?

[1]

ii Suppose the 15 months from the time of clinic's opening is divided into n equal sub intervals. How long will each of these sub intervals be ?

[1]

iii How many new patients will be admitted to the clinic during any of the sub intervals discussed in **2c ii**?

[1]

We can write out the sub intervals as follows:

$$\left[0, \frac{15}{n}\right], \left[\frac{15}{n}, \frac{30}{n}\right] \ldots\ldots\ldots\ldots \left[\frac{15(j-1)}{n}, \frac{15j}{n}\right] \ldots\ldots\ldots\ldots \left[\frac{15(n-1)}{n}, 15\right]$$

iv

1 Write an expression for m_j, the middle of the j$^{\text{th}}$ sub interval after the opening of the clinic.

[2]

2 Write an expression, involving m_j, to express the time that will pass from the middle of the j$^{\text{th}}$ sub interval until the end of the 15 months.

[1]

v Suppose that all patients admitted in the j$^{\text{th}}$ sub interval are admitted in the middle of this sub interval. Write down an expression for the fraction of these patients that are still being treated at the end of 15 months after the clinic starts operations.

[1]

vi Using what you have found in **2c i** and **2c v** explain why the number of patients receiving treatment at the end of 15 months is approximately

$$300e^{-0.75} + \sum_{j=0}^{n-1} 10e^{\frac{m_j - 15}{20}} \frac{15}{n} \quad \text{where } m_j \text{ is defined in } \textbf{2c iv}$$

[1]

vii Let $n \to \infty$ and using the resulting integral find the number of patients still being treated at the clinic.

[3]

Question 2.

a A comet orbits around a pair of planets which are located in the same plane at points (-d,0) and (d,0) in a Cartesian system of coordinates. The comet moves so that the sum of its distances from the two planets has a constant value of 2R.

i Write down an expression for the distance of the comet from the planet at (d,0) when the position of the comet is (x,y).

[1]

ii Write the equation of the path of the comet in terms of x, y , d and R. Do not simplify at this stage.

[1]

iii Prove that when the comet crosses the straight line joining the two planets it crosses the line at the point (R,0) or the point (-R,0).

[3]

iv Prove that when the comet is equidistant from the two planets that $x = 0$ and $y^2 = R^2 - d^2$ where (x,y) are the coordinates of the comet's position.

[2]

v <u>Prove</u> that the equation of the comet's path can be written as

$$\frac{x^2}{R^2} + \frac{y^2}{R^2 - d^2} = 1.$$

You may not assume the equation of the path has the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

[4]

b An ellipse has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.

i Sketch the ellipse.

The point C, (c,0) , is such that $c > a$ and from the point C tangents to the ellipse are drawn.

On the same diagram sketch the point C and the two tangents.

[1]

ii One of the tangents meets the ellipse at the point (m,n) where $n > 0$.

Prove that the gradient of the tangent at the point (m,n) is $\frac{-b^2 m}{a^2 n}$.

[2]

iii Using 2(b)(ii) or otherwise prove that $m = \frac{a^2}{c}$.

[3]

iv Prove that $n = \frac{b}{c}\sqrt{(c^2 - a^2)}$.

[2]

v Discuss what happens when $c \to a$.

[1]

Question 3.

a Marsha and Hemi believe they have proved that if you inscribe a rectangle in a circle (so that both of its diagonals are diameters) then the maximum area will occur when the rectangle is a square. Determine if they are correct by using calculus.

[10]

b Sione, a friend of theirs, uses a similar method and finds that the maximum volume of a box whose eight corners are on the surface of a sphere will occur when the box is a cube. Assuming this result, if the radius of the sphere is R find the volume of this cube.

[5]

c He also notices that if you connect the centre of the sphere to the four corners of the base of the cube then a square-based pyramid is formed. Find the tangent of the angle made by the lines joining the centre of the cube to the ends of a diagonal of the base.

[5]

Question 4.

a The piriform is the curve defined by the equation $16y^2 = x^3(8-x)$ where $x \geq 0$.

It has the parametric equations
$$x = 4(1+\sin\theta)$$
$$y = 4(1+\sin\theta)\cos\theta .$$

i By solving the differential equation

$$\frac{dy}{dx} = \frac{x^2(6-x)}{8y}, \quad y=0 \text{ when } x=0$$

show the piriform is the solution .

[3]

ii Explain why the piriform is symmetric about the x-axis.

[1]

iii Show that
$$x = 4(1+\sin\theta)$$
$$y = 4(1+\sin\theta)\cos\theta$$
are parametric equations for the piriform.

[3]

iv Find $\dfrac{dy}{dx}$ in terms of θ.

[4]

v Substitute to show $\dfrac{dy}{dx} = 0$ when $\theta = \dfrac{\pi}{6}$.

[1]

vi Hence find the co-ordinates of the corresponding stationary point.

[2]

vii What is the gradient at $x = 8$?

[1]

viii Sketch the curve. [Assume the gradient is 0 at $x = 0$]

[3]

b Find the volume of revolution obtained by rotating the piriform about the x-axis.

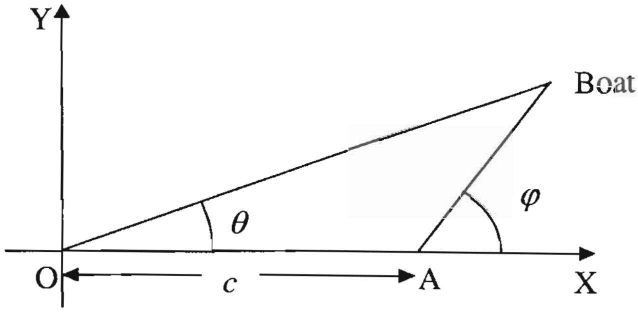
[2]

Question 5.

a Show that, for any angle θ, $\cot(180-\theta) = -\cot\theta$.

[1]

b The diagram shows a boat at sea observed from two points, O and A, distant c apart, on the beach. The line from O to the boat makes an angle θ with the line OA and the line from A to the boat makes an angle φ, as shown. A coordinate system is added as shown.



i By drawing the perpendicular from the boat to the beach and considering the right-angled triangles so formed, or otherwise

1 Show that the x-coordinate of the boat is $x = \dfrac{c\tan\varphi}{\tan\varphi - \tan\theta}$. [4]

2 Show the x-coordinate of the boat can be written as $x = \dfrac{c\sin\varphi\cos\theta}{\sin(\varphi-\theta)}$.

Find the y-coordinate. [4]

3 Show that the boat is distant $D = \dfrac{c\sin\varphi}{\sin(\varphi-\theta)}$ from O. [2]

ii Now let φ be any fixed angle and let θ vary, keeping c constant.

1 Use calculus to find the minimum value of D. You may assume that the value you find is a minimum.

[2]

2 For different values of φ, the position of the boat at which D is a minimum varies. Describe the curve along which these minimum positions lie and give a reason for your choice of curve.

[2]

c

i If $L_n = \lim\limits_{\theta\to 0}\dfrac{\sin n\theta}{\sin\theta}$, what is L_{n-1}?

[1]

ii By writing $n\theta = (n-1)\theta + \theta$, show that $L_n = 1 + L_{n-1}$.

[2]

iii Hence or otherwise explain why $\lim\limits_{\theta\to 0}\dfrac{\sin n\theta}{\sin\theta} = n$.

[2]

Question 6.

a

i Given $P(z)=(z^2-2z+4)(z-i)$, find the roots of $P(z)=0$.

[2]

ii Show that if $P(z)$ is a polynomial with complex roots a and $\bar{a}$, then $z^2-2z\mathrm{Re}(a)+|a|^2$ is a factor of $P(z)$. [Note: $\bar{a}$ is the conjugate of a and $\mathrm{Re}(a)$ is the real part of a.]

[3]

b Given that $z=\cos\theta+i\sin\theta$,

i Use De Moivre's theorem to show that $z^n+z^{-n}=2\cos n\theta$.

[2]

ii Write down the corresponding result for z^n-z^{-n}.

[1]

c Consider the equation $z^{2n}=1$.

i Show that the roots of this equation can be written in the form -1, 1, $\cos\frac{k\pi}{n}\pm i\sin\frac{k\pi}{n}$ $\quad(k=1,\ldots\ldots,n-1)$.

[3]

ii Explain why the result in **ci** above implies that z^2-1 is a factor of $z^{2n}-1$.

[1]

iii Using the result in **aii**, show

$$z^{2n}-1=(z^2-1)\left(z^2-2z\cos\left(\frac{\pi}{n}\right)+1\right)\left(z^2-2z\cos\left(\frac{2\pi}{n}\right)+1\right)\ldots\ldots\left(z^2-2z\cos\left(\frac{(n-1)\pi}{n}\right)+1\right).$$

and hence that

$$z^n-z^{-n}=(z-z^{-1})\left(z+z^{-1}-2\cos\left(\frac{\pi}{n}\right)\right)\left(z+z^{-1}-2\cos\left(\frac{2\pi}{n}\right)\right)\ldots\ldots\left(z+z^{-1}-2\cos\left(\frac{(n-1)\pi}{n}\right)\right).$$

[3]

iv Using the results in **b**, show

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\left(\cos\theta - \cos\left(\frac{\pi}{n}\right)\right)\left(\cos\theta - \cos\left(\frac{2\pi}{n}\right)\right)\ldots\ldots\left(\cos\theta - \cos\left(\frac{(n-1)\pi}{n}\right)\right).$$

[2]

v Hence, assuming the result proved in Question 5c that $\lim\limits_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = n$,

show that for $n > 1$

$$n = 2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\ldots\sin\left(\frac{(n-1)\pi}{n}\right).$$

[3]