NZEST SCHOLARSHIP EXAMINATION 2001

MATHEMATICS WITH CALCULUS

TO ISSUE: STANDARD ANSWER BOOK

Tuesday 13 November 2001, 2pm

Time allowed: Three hours

Notes for Candidates

1 This examination consists of **SIX** questions, each worth 20 marks

2 Please attempt **ALL SIX** questions

3 You will get credit for every question (or part question) you do

4 Please show all working

5 Calculators approved by NZQA may be used.

6 Tables of formulae are supplied as a detachable centrefold sheet.

Question 1 .

a The depth, d, of the tide in a bay is modelled by the formula
$d = A + B\cos(Ct + D)$
where A,B,C and D are constants, d is measured in metres and t in hours. The time between successive high tides is 14 hours. The maximum depth of the tide in the bay is 5m and the minimum depth is 1m. Initially the depth is 4m and the tide is coming in.

Find A, B,C and D. [7]

b A boat, distance d from the shore makes an angle of elevation α with a point halfway up a vertical cliff on the shore. The boat sails a further distance k directly away from the shore so that its distance from the shore is now $d + k$. The boat now makes an angle of elevation, β, with the top of the cliff.

Prove that $k = d\left(\dfrac{2\sin\alpha\cos\beta}{\sin\beta\cos\alpha} - 1\right)$. [5]

c It is desired to find the volume of the shape obtained by revolving $y = 2\sin x,\ 0 \le x \le \pi$ about the X axis. This is found in three ways, first by evaluating the correct formula using the trapezium rule with 4 sub divisions, secondly by evaluating the correct formula using Simpson's rule with 4 sub-divisions and finally by integration.

Comment on the accuracy of the two numerical calculations. [8]

Question 2 .

a A satellite moves in space in a rectangular coordinate system so that its distance from the point $(a,0)$ is always equal to its distance from the line $x=b$.

i If the position of the satellite is given by (x,y) write down a mathematical expression for its distance d_1 from the point $(a,0)$. [1]

ii Write down a mathematical expression for the distance d_2 of the satellite, still presumed to be at (x,y), from the line $x=b$. [1]

iii Using the fact that $d_1 = d_2$ and that the satellite crosses the X axis at (0, 0) find the relationship between a and b. [3]

iv Prove the equation which governs all points (x,y) lying on the path of the satellite is $y^2 = 4ax$. [3]

v A line is drawn at right angles to the X axis through $(a,0)$. Find the coordinates of the points where this line cuts the path of the satellite. [2]

.

b Two parabolas $y^2 = 4gx$ and $y^2 = -4h(x-c)$ meet in such a way that the tangents to each parabola at both points of intersection are at right angles.

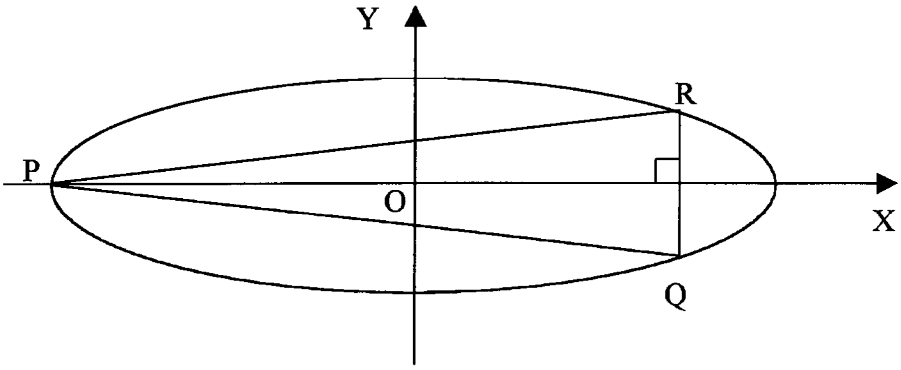
i Sketch this situation on a diagram. [2]

ii By considering the gradients of the tangents at the points where the parabolas intersect, find the coordinates of the points of intersection of the two parabolas in terms of g and h. [6]

iii Prove $c = g + h$. [2]

Question 3 .

a Marsha and Hemi are arguing about the isosceles triangle PQR of largest area which can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the way shown in the diagram below.



Marsha believes that the maximum area will be when QR lies on the Y axis while Hemi disagrees with this.

Who is correct ? Find this out, showing all mathematical and other arguments.

[11]

b Their friend Sione believes they are wasting their time and should be investigating three dimensional ellipsoidal shapes with a circular cross-section.

He has read that such a shape would have the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1$.

He uses this equation to find the maximum volume of a cone which can be embedded in this ellipsoid so that a slice along the major axis would appear like the diagram above.

He uses the formula for the volume of a cone

$V=\frac{1}{3}\pi r^2 h$ where r is the base radius of the cone and h its height.

Find the maximum volume of this cone using Sione's method or any other.

[9]

Question 4 .

a By considering $(\text{cis}\varphi)^3 = (\cos\varphi + i\sin\varphi)^3$, obtain the formula for $\sin 3\varphi$ in terms of $\sin\varphi$.
(Note: You must use the above method.) [5]

b Let n be a positive integer.

i Find the solutions to $z^n = 1$. [2]

ii Hence show the solutions to $(1-z)^n = z^n$, can be written in the form
$z = \frac{1}{2} - \frac{i}{2}\tan\frac{k\pi}{n}, \quad (k = 0,1,\ldots\ldots,n-1)$. [5]

c Show e is irrational as follows:

i Show that $\frac{1}{n+k} < \frac{1}{n+1}$, where n and k are positive integers and $k > 1$. [1]

ii Given $1 + r + r^2 + r^3 + \ldots\ldots\ldots = \frac{1}{1-r}$ for $|r| < 1$,

show that $\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ldots\ldots\ldots\ldots < \frac{1}{n}$.

[3]

iii Suppose e is a rational number so we can write
$\frac{m}{n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots\ldots + \frac{1}{n!} + \frac{1}{(n+1)!} + \ldots\ldots\ldots$ where m, n are positive integers.

By multiplying each side by $n!$, and using the result in **4c ii,** show this leads to a contradiction.

[4]

Question 5 .

a Evaluate $\int \frac{\sqrt{1-y^2}}{y} dy$ as follows:

i Prove $\sin\varphi = \dfrac{2\tan\frac{\varphi}{2}}{\sec^2\frac{\varphi}{2}}$. [2]

ii Hence prove $\int \operatorname{cosec}\varphi \, d\varphi = \log_e\left|\tan\frac{\varphi}{2}\right| + c$. [3]

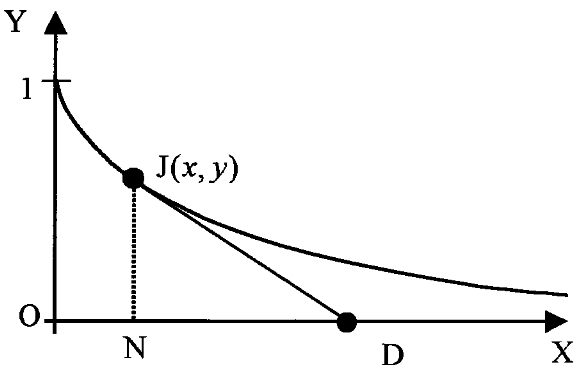
iii By making the substitution $y = \sin\varphi$ show that

$$\int \frac{\sqrt{1-y^2}}{y} dy = \sqrt{1-y^2} + \log_e\left|\frac{y}{1+\sqrt{1-y^2}}\right| + c.$$ [6]

b Find the coordinates of the point where a line with gradient m, passing through the point (a, b), cuts the x-axis. [1]

c John is taking Marsha's dog Rufus for a walk in a park when Rufus sees a rabbit at right angles to their path. The dog immediately chases after the rabbit dragging John with it. The length of the dog's lead is 1m. The problem is to describe John's path if the dog continues to run in a straight line.

Model the problem as shown by the graph. Initially John is at 1 on the y-axis while the dog is at O. The dog then runs in the direction of the positive x-axis. J represents the position of John at some time and D the position of the dog at that time. JD is then the dog's lead.



i Since John is always being pulled towards the dog, the dog's lead lies on the tangent to the curve at J. What are the coordinates of D in terms of x, y and y', the gradient of the curve at (x, y)? (Use the result in **5 b**.) [1]

ii Explain why the differential equation describing John's path can be written as $\frac{y}{y'} = -\sqrt{1-y^2}$ where $y' = \frac{dy}{dx}$. [2]

iii Solve this equation by separating variables to obtain the equation of John's path in the form $x = f(y)$. [4]

iv How far has John travelled horizontally, when he is 1mm above the x-axis. [1]

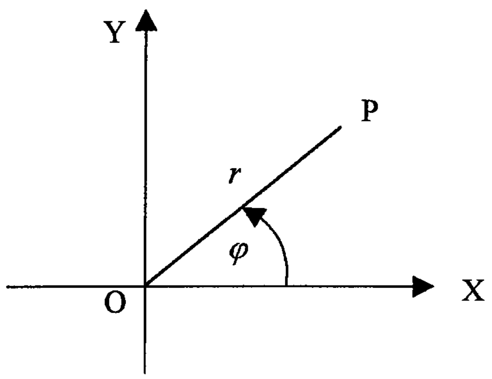
Question 6 .

a Assume $\frac{\sin\varphi}{\varphi} \approx 1$, for small values of φ. (You have probably used some equivalent form in class to find the derivative of $\sin x$.)

i Starting from the formula for $\cos(A+B)$, write $\cos\varphi$ in terms of $\sin\frac{\varphi}{2}$. [2]

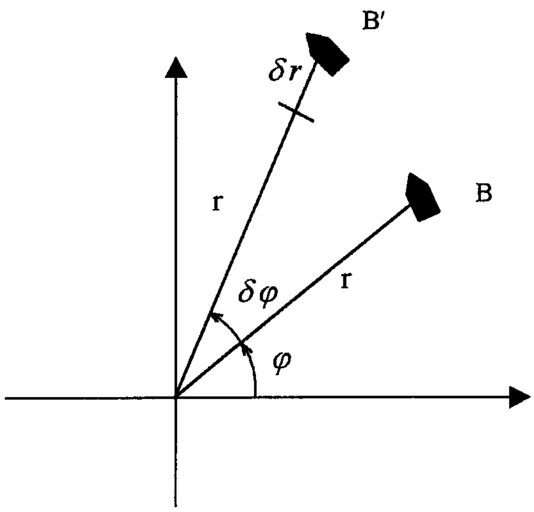
ii Hence show $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$ for small values of φ.
($\approx$ means approximately equal to) [2]

b Polar coordinates (r,φ) are an alternative way of expressing the position of a point to the usual Cartesian (x,y) coordinates. Given a coordinate system, the distance of the point P from the origin O specifies r and the angle between the positive x-axis and the line joining the origin to P specifies φ.



Write down the Cartesian (x,y) coordinates of P. [1]

c Within a given coordinate system, a boat moves from position B to position B′ as shown in the diagram. The polar coordinates of B are (r,φ) and the polar coordinates of B′ are $(r+\delta r, \varphi + \delta\varphi)$.



i Explain why the distance δs from B to B′ is given by

$$\delta s^2 = (r+\delta r)^2 + r^2 - 2r(r+\delta r)\cos(\delta\varphi).$$ [1]

ii If δr and $\delta\varphi$ are both very small, use the approximation $\cos(\delta\varphi) \approx 1 - \frac{(\delta\varphi)^2}{2}$, found earlier in this paper, to show $\delta s^2 \approx \delta r^2 + r^2(\delta\varphi)^2$. Explain why, in obtaining this formula, you could ignore the term containing $\delta r(\delta\varphi)^2$.

[3]

iii Read the following, which you will need shortly. There is no working required for this part of the question.

Dividing the expression by $(\delta t)^2$, where δt is the time taken to travel from B to B′, gives

$$\left(\frac{\delta s}{\delta t}\right)^2 \approx \left(\frac{\delta r}{\delta t}\right)^2 + r^2\left(\frac{\delta\varphi}{\delta t}\right)^2.$$

Letting $\delta t \to 0$ gives

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2$$ where v is the speed of the boat.

d A police boat travelling at 50 km/hr is chasing a boat travelling at 30 km/hr. These speeds are the maximum speeds for each boat. When the police boat is some distance from its target, heavy fog comes down and the other boat can no longer be seen. The origin of an axis system is taken to be the position of the other boat at this time. The police assume that the other boat changes direction but continues to travel in a straight line at 30 km/hr. They estimate it will have travelled 1km in the time it would take them to reach the origin. The problem is to find what path the police boat should take to be best able to intercept the other boat.

Let (r, φ) be the polar coordinates of the police boat t hrs after losing sight of the other boat.

i Explain why, the police boat should choose its path so that its distance from the origin is given by $r = 1 + 30t$. [1]

ii Using the expression $v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2$, show $r^2\left(\frac{d\varphi}{dr}\right)^2 = \frac{16}{9}$. [4]

iii Hence find r in terms of φ to obtain the path the police boat should follow. [3]

iv Give a description of the general shape of this path in words. [1]

v What would be the best strategy for the other boat to avoid capture? You must explain your strategy in terms of your answer to **6d iv** above. [2]

END OF QUESTION PAPER

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NEW ZEALAND EDUCATION & SCHOLARSHIP TRUST

MATHEMATICS WITH CALCULUS

2001 FORMULAE AND TABLES

Carefully remove these pages which contain useful formulae and tables.

You may keep the formulae and tables at the end of the examination.

Acknowledgement:
These formulae and tables are reprinted with the permission
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MATHEMATICS WITH CALCULUS – USEFUL FORMULAE AND TABLES

ALGEBRA

Arithmetic Sequences and Series

$t_n = a + (n-1)d$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Geometric Sequences and Series

$t_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1$

$S_\infty = \frac{a}{1-r} \quad$ for $|r| < 1$

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!\,r!}$

Some values of $\binom{n}{r}$ are given in the table below.

Quadratic Equations

If $ax^2 + bx + c = 0$

then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Logarithms

$\ln x = \log_e x$

If $y = b^x$ then $\log_b y = x$

If $y = e^x$ then $x = \ln y = \log_e y$

$\log_b x + \log_b y = \log_b xy$

$\log_b x - \log_b y = \log_b \frac{x}{y}$

$\log_b x^n = n \log_b x$

Binomial Coefficients

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66
13	1	13	78	286	715	1287	1716	1716	1287	715	286
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003

Complex Numbers

$z = x + iy$

$\quad = r(\cos\theta + i\sin\theta)$

$\quad = r\,\mathrm{cis}\theta$

$\bar{z} = x - iy$

$|z| = \sqrt{(x^2 + y^2)} = r$

$\arg z = \theta$ where $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$

De Moivre's Theorem: If n is any integer then $(r\,\mathrm{cis}\theta)^n = r^n\mathrm{cis}(n\theta)$

CALCULUS

Differentiation

$y=f(x)$	$\frac{dy}{dx}=f'(x)$
c	0
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^{ax}	ae^{ax}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$

Integration

$f(x)$	$\int f(x)\,dx$
k	$kx+c$
x^n	$\frac{x^{n+1}}{n+1}+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$

Product Rule

$(f.g)'=f.g'+g.f'$

or if $y=uv$ then $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

Quotient Rule

$\left(\frac{f}{g}\right)'=\frac{g.f'-f.g'}{g^2}$

or if $y=\frac{u}{v}$ then $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

Composite Function or Chain Rule

If $h(x)=f(g(x))$ then $h'(x)=f'(g(x)).g'(x)$

or if $y=f(u)$ and $u=g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$

COORDINATE GEOMETRY

Straight Line

Gradient $m=\frac{y_2-y_1}{x_2-x_1}$

Equation $y-y_1=m(x-x_1)$

Distance $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Midpoint $M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Circle

$x^2+y^2=r^2$ or $(r\cos\theta, r\sin\theta)$

has centre $(0,0)$ and radius r

$(x-a)^2+(y-b)^2=r^2$

has centre (a,b) and radius r

Parabola

$y^2=4ax$ or $(at^2, 2at)$

Ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $(a\cos\theta, b\sin\theta)$

Hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $(a\sec\theta, b\tan\theta)$

asymptotes $y=\pm\frac{b}{a}x$

MEASUREMENT

Triangle

area $=\frac{1}{2}ab\sin C$

Trapezium

area $=\frac{1}{2}(a+b)h$

Circle

circumference $=2\pi r$

area $=\pi r^2$

Sector

area $=\frac{1}{2}r^2\theta$

arc length $=r\theta$

Cylinder

volume $=\pi r^2 h$

curved surface area $=2\pi rh$

Cone

volume $=\frac{1}{3}\pi r^2 h$

curved surface area $=\pi rl$ where $l=$ slant height

Sphere

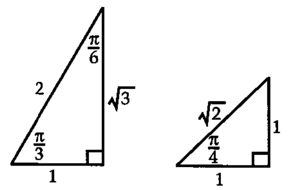
volume $=\frac{4}{3}\pi r^3$

surface area $=4\pi r^2$

(Turn over

TRIGONOMETRY

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$



Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine Rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Identities

$$\cos^2\theta + \sin^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Compound Angles

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Double Angles

$$\sin 2A = 2\sin A\cos A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

Products

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

Sums

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$
$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$
$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$
$$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

General Solutions

If $\sin\theta = \sin\alpha$ then $\theta = n\pi + (-1)^n\alpha$

If $\cos\theta = \cos\alpha$ then $\theta = 2n\pi \pm \alpha$

If $\tan\theta = \tan\alpha$ then $\theta = n\pi + \alpha$

where n is any integer

NUMERICAL METHODS

Trapezium Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h\left[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

where $h = \dfrac{b-a}{n}$ and $y_r = f(x_r)$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{3}h\left[y_0 + y_n + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right]$$

where $h = \dfrac{b-a}{n}$, $y_r = f(x_r)$ and n is even.