NZEST SCHOLARSHIP EXAMINATION 2002

MATHEMATICS WITH CALCULUS

TO ISSUE: STANDARD ANSWER BOOK

Tuesday 12 November 2002, 2pm

Time allowed: Three hours

Notes for Candidates

1 This examination consists of **SIX** questions, each worth 20 marks

2 Please attempt **ALL SIX** questions

3 You will get credit for every question (or part question) you do

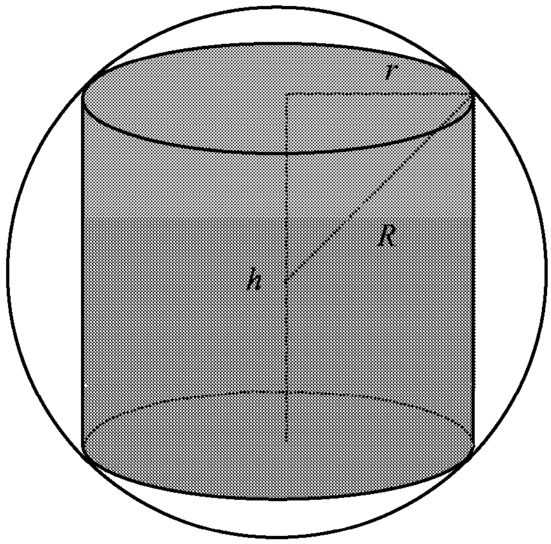
4 <u>Please show all working</u>

5 Calculators approved by NZQA may be used.

6 Tables of formulae are supplied as a detachable centrefold sheet.

Question One

a. A reactor in the form of a cylinder of radius r and height h, is contained in a spherical pressure vessel of radius R, as shown.



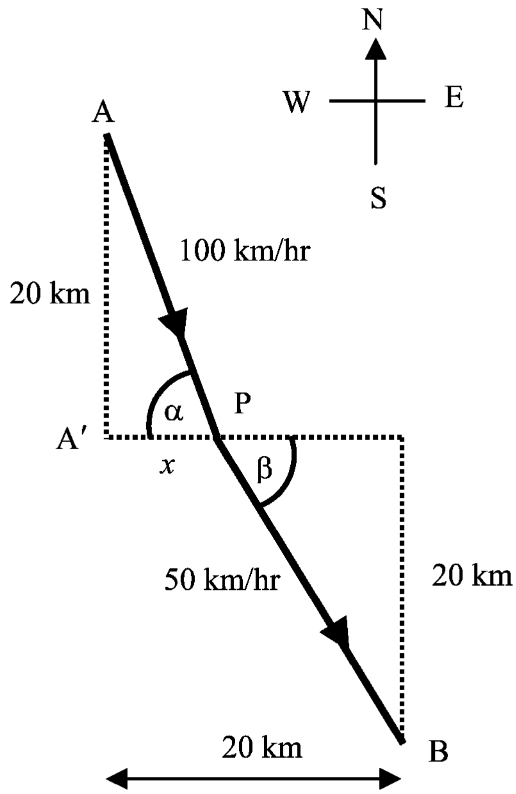
(i) If R is fixed, find the values of h and r corresponding to the cylinder having maximal volume, explaining why these values correspond to maximum volume. [5]

(ii) Find the ratio of the volume of the maximal cylinder to that of the sphere. [2]

(The volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$ and of a cylinder of radius r and height h is $V = \pi r^2 h$)

b. A and B are two places situated on a flat, featureless desert plane. A car can travel from point A to point B anywhere across this plane. A and B are 40km apart on the North/South axis and 20km apart on the West/East axis. A car travelling from A to B can travel at 100km/hr north of the West/ East axis, but only at 50km/hr south of this axis, owing to the ground being softer.

The car drives on a straight line from A to P, a point on the East/West axis. It then changes direction and drives on a straight line from P to B. The angles made with the East/West axis are α and β as shown on the diagram.



(i) If the length of A′P is x, give $\cos\alpha$ and $\cos\beta$ in terms of x. [2]

(ii) Show that, if the journey takes minimum time

$$\cos\alpha = 2\cos\beta$$

[11]

[You can assume the time is a minimum and not a maximum]

Question Two

Mary is about to make her first parachute jump. Together with her gear, she weighs m kg. The differential equation which describes her speed of fall, v, is

$$m\frac{dv}{dt} = 10m - kv$$

where k is a constant relating to air resistance.

Mary's initial speed of fall is zero. Therefore, when $t = 0$, $v = 0$.

a.

(i) By separating variables and integrating, show that the solution can be written in the form

$$A\left(1 - e^{Bt}\right)$$

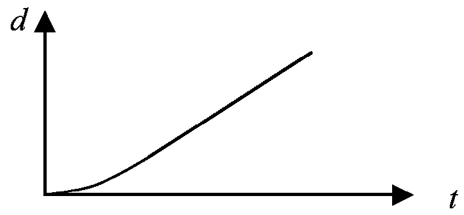
and find what A and B are. [9]

(ii) Sketch the graph of the solution to **a.**(i). It shows a very interesting feature relating to Mary's speed. Write a sentence describing this feature and how it relates to Mary's actual jump. [3]

b.

(i) Find the distance Mary has fallen as a function of time. [5]

(ii) Here is a sketch of the distance/time graph.



Describe how the features of this graph arise from the solution and relate this graph to the speed/time graph. [3]

Question Three

a. In very many applications, especially in statistics and probability, it is important to be able to approximate n!, where n is a large integer.

(i) By first differentiating $x\log_e x - x$, evaluate $\int_1^n \log_e x \, dx$. [2]

(ii) If n is an integer, use the trapezium rule with an interval width of 1 to prove

$$\int_1^n \log_e x \, dx \approx \log_e n! - \frac{1}{2}\log_e n$$ [3]

b.

(i) Show $\int_1^n \log_e x \, dx$ can be written in the form $\log_e\left(n^n e^{1-n}\right)$. [2]

(ii) On the assumption that the results of **a.(ii)** and **b.(i)** are approximately the same, show that $n! \approx n^{n+1/2} e^{1-n}$.

[3]

Making this argument more precise gives Stirling's famous formula:

$$\sqrt{2\pi}\; n^{n+1/2} e^{-n} < n! < \sqrt{2\pi}\; n^{n+1/2} e^{-n}\left(1+\frac{1}{4n}\right)$$

We will obtain a weaker result.

c. We now approximate the area by trapezia in a different way to that given by the Trapezium Rule. This is shown on the next page.

Consider the trapezium of base length 1, shown in fig 1.
Its area is base × average height $= h$

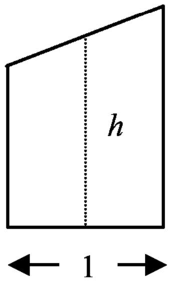


fig 1

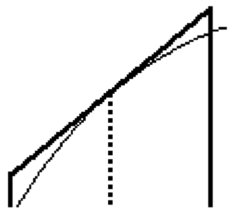
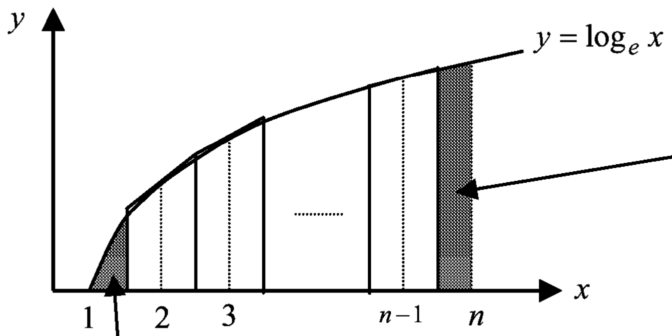


fig 2

For each x-value = 2,3,4,.......,n-1 draw the tangent to the curve as shown in fig 2 so that the vertical line through this x-value forms the centre of a trapezium of width 1.

The following diagram shows the area under $y = \log_e x$ approximated by trapezia of width 1 about 2,3,....,n-1 and an area at each end. The instructions on the diagram indicate how the end areas are to be calculated.



Approximate this last area, between $n - \frac{1}{2}$ and n, by a rectangle of width $\frac{1}{2}$ and height $\log_e n$

As the midpoint of the interval between 1 and $\frac{3}{2}$ is $\frac{5}{4}$, approximate this first area, by a trapezium of width $\frac{1}{2}$ and average height $\log_e \frac{5}{4}$.

(i) Show that the difference in the area approximated this way and approximated as in **a.**(ii) is $\frac{1}{2}\log_e \frac{5}{4}$. [4]

(ii) Use graphical considerations to explain in words why $\int_1^n \log_e x \, dx$ is greater than the approximation of **a(ii)** but less than that of **c(i)**. [1]

Hence show that $\sqrt{\frac{4}{5}}e^{1-n}n^{n+1/2} < n! < e^{1-n}n^{n+1/2}$ [5]

Question Four

a. Starting from the formulae for $\sin 2x$ and $\cos 2x$:

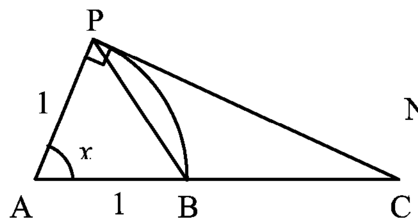
(i) Show that $\sin\dfrac{\alpha}{2^n}\cos\dfrac{\alpha}{2^n} = \dfrac{1}{2}\sin\dfrac{\alpha}{2^{n-1}}$ [1]

(ii) Show that $\sqrt{\dfrac{1}{2}+\dfrac{1}{2}\cos\alpha} = \cos\dfrac{\alpha}{2}$ [1]

b.

(i) By considering the area of the triangle APB, the circular sector APB and the triangle APC, show that $1 \le \dfrac{x}{\sin x} \le \dfrac{1}{\cos x}$, where x is in radians.

[2]



Note: The radius of the sector is 1

(ii) Hence, or otherwise, show that $\lim\limits_{n\to\infty}\left(\dfrac{1/2^n}{\sin(\alpha/2^n)}\right) = \dfrac{1}{\alpha}$ [3]

c. Using the results in **a.** and **b.** above:

(i) Prove that $\cos\dfrac{\alpha}{2}\cos\dfrac{\alpha}{4}\cos\dfrac{\alpha}{8}.............\cos\dfrac{\alpha}{2^{n-1}}\cos\dfrac{\alpha}{2^n} = \dfrac{1}{2^n}\dfrac{\sin\alpha}{\left(\sin\dfrac{\alpha}{2^n}\right)}$

Hint: Start by considering

$$\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\frac{\alpha}{8}.............\cos\frac{\alpha}{2^{n-1}}\cos\frac{\alpha}{2^n}\sin\frac{\alpha}{2^n}$$

and combining the last two terms. [5]

(ii) By first taking the limit as $n\to\infty$ of both sides, show that

$$\alpha = \frac{\sin\alpha}{\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\frac{\alpha}{8}.............}$$

[4]

(iii) By setting $\alpha = \dfrac{\pi}{2}$, show that

$$\frac{\pi}{2} = \frac{1}{\left(\sqrt{\frac{1}{2}}\right)\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\right)\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\right)............}$$

[4]

Note: This remarkable representation was discovered by Francois Viete at the end of the 16th century. It was the first numerically exact expression of π and also the first to express π as an infinite product.

Question Five

In recent years, scientists have become concerned about the possibility of the Earth being struck by an asteroid. Asteroids orbit the sun on elliptical paths with the Sun at one focus. One proposal is to park a rocket at one of the Lagrange points, where the gravitational fields of the Earth and Sun balance out. When an asteroid approaches, the rocket can be fired at it along a hyperbolic path, to strike at right angles and deflect it from the Earth. We will look at a simpler version of this problem, but one day you might use this theory to save the Earth.

Assume that, for some system of coordinates, the asteroid is travelling along an ellipse and the rocket along the hyperbola represented respectively by

$$\frac{x^2}{a^2}+\frac{y^2}{A^2}=1 \text{ and } \frac{x^2}{b^2}-\frac{y^2}{B^2}=1$$

a. Given that $b < a$, sketch the two curves, giving the intercepts with the axes. [1]

b. Given that the curves intersect at (x_0, y_0)

(i) By considering gradients, find a condition in terms of a, A, b, B, x_0, y_0 for the two curves to intersect at right angles. [4]

(ii) Hence or otherwise, show that if $A^2 + B^2 = a^2 - b^2$, then the curves intersect at right angles. In other words, assuming the condition, show they meet at right angles. [6]

c. For the ellipse the eccentricity is e (<1), the foci $(\pm ae, 0)$ and $A^2 = a^2(1-e^2)$.
For the hyperbola the eccentricity is f (>1), the foci $(\pm bf, 0)$ and $B^2 = b^2(f^2-1)$.

For the rest of this question (parts **c.** and **d.**) assume the two curves are confocal. This means that they both have the same foci, and so $ae = bf$.

Using **b.**(ii) or otherwise, show that these curves meet at right angles. [3]

d.

(i) Explain why the approximate path of the rocket is a straight line if the distances are large enough. [1]

(ii) Find the equation of this approximate path in terms of a, e, b given the intersection occurs in the positive quadrant ($x > 0,\ y > 0$). [3]

(iii) If $a = 2b$, find the coordinates of the intersection of this straight line with the ellipse in terms of A and e. [2]

Question Six

a. This part uses only real numbers.

(i) By expanding $\left(x^m + \dfrac{1}{x^m}\right)\left(x^n + \dfrac{1}{x^n}\right)$ obtain a formula for $x^{m+n} + \dfrac{1}{x^{m+n}}$ and hence for $x^{m+1} + \dfrac{1}{x^{m+1}}$ and $x^{2m} + \dfrac{1}{x^{2m}}$ [3]

(ii) Hence, or otherwise, evaluate $x^{24} + \dfrac{1}{x^{24}}$, given $x + \dfrac{1}{x} = a$.
You do not need to simplify your answer by expanding any brackets. [3]

b.

(i) Expand $(a+b)^2(a-b)^3$ by first rewriting the product to include a power of an $(a^2 - b^2)$ term. [2]

(ii) Let $z = \cos\theta + i\sin\theta$.

(1) Use De Moivre's Theorem to show that $\dfrac{1}{z} = \cos\theta - i\sin\theta$ [1]

(2) Show $2\cos n\theta = z^n + \dfrac{1}{z^n}$ [2]

(3) Write down the corresponding result for $2i\sin n\theta$ [1]

(iii)

(1) If $z = \cos\theta + i\sin\theta$, use the results in **b.(i)** and **b.(ii)**, with $n = 1$, to express $(2\cos\theta)^2(2i\sin\theta)^3$ as a sum of terms of the form $Ai\sin k\theta$, where A is a real number and $k \geq 0$ is an integer, and hence express $\cos^2\theta\sin^3\theta$ as a sum of multiples of θ. [5]

(2) Imogen makes the hypothesis that all such products of powers of $\sin\theta$ and $\cos\theta$ can be expanded as sums of sines of multiples of θ.

Jake claims Imogen is wrong and produces the counter example $\cos^2\theta\sin^0\theta$. Explain why $\cos^2\theta\sin^0\theta$ cannot be expanded in this form. [1]

(3) Jake goes on to say that there are two possibilities for the form of the expansion and he knows the condition determining which type will occur. State what the two possible forms of the expansion are and what condition determines which occurs. Give reasons. [2]

END OF EXAM PAPER

Please enter the information required
on the back flap of your answer booklet.

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MATHEMATICS WITH CALCULUS

2002 FORMULAE AND TABLES

Carefully remove these pages which contain useful formulae and tables.

You may keep the formulae and tables at the end of the examination.

Acknowledgement:
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MATHEMATICS WITH CALCULUS – USEFUL FORMULAE AND TABLES

ALGEBRA

Quadratic Formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Arithmetic Sequences and Series

$t_n = a + (n-1)d$

$S_n = \dfrac{n}{2}\left[2a + (n-1)d\right]$

Logarithms

$y = \log_b x \Leftrightarrow x = b^y$

$\log_b(xy) = \log_b x + \log_b y$

$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

$\log_b\left(x^n\right) = n\log_b x$

$\log_b x = \dfrac{\log_a x}{\log_a b}$

Geometric Sequences and Series

$t_n = ar^{n-1}$

$S_n = \dfrac{a\left(1-r^n\right)}{1-r} \qquad r \neq 1$

$S_\infty = \dfrac{a}{1-r} \qquad \text{for } |r| < 1$

Complex Numbers

$z = x + iy = r\,\text{cis}\,\theta = r(\cos\theta + i\sin\theta)$

$\bar{z} = x - iy = r\,\text{cis}(-\theta) = r(\cos\theta - i\sin\theta)$

$r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$

$\theta = \arg(z)$ where $\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$

$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$ for integer n (De Moivre's Theorem)

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

where $\dbinom{n}{r} = {}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

Some values of $\dbinom{n}{r}$ are given in the table below.

Binomial Coefficients

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66
13	1	13	78	286	715	1287	1716	1716	1287	715	286
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003

CALCULUS

Differentiation

y	$\frac{dy}{dx}$
c	0
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^{ax}	ae^{ax}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$

Integration

y	$\int y\,dx$
k	$kx+c$
x^n	$\frac{1}{n+1}x^{n+1}+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
$\frac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert+c$

Product Rule

$(f.g)' = f.g' + g.f'$

or if $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient Rule

$\left(\frac{f}{g}\right)' = \frac{g.f' - f.g'}{g^2}$

or if $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Volume of Revolution

$y = f(x)$ between $x = a$ and $x = b$

rotated about the x - axis.

Volume $= \int_a^b \pi y^2\,dx$

Composite Function or Chain Rule

$(f(g))' = f'(g).g'$

or if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

COORDINATE GEOMETRY

Straight Line

Gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$

Equation $y - y_1 = m(x - x_1)$

Distance $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Midpoint $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Circle

$x^2 + y^2 = r^2$ or $(r\cos\theta, r\sin\theta)$

has centre $(0,0)$ and radius r

$(x-a)^2 + (y-b)^2 = r^2$

has centre (a,b) and radius r

Parabola

$y^2 = 4ax$ or $(at^2, 2at)$

Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $(a\cos\theta, b\sin\theta)$

Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $(a\sec\theta, b\tan\theta)$

asymptotes $y = \pm\frac{b}{a}x$

MEASUREMENT

Triangle

area $= \frac{1}{2}ab\sin C$

Trapezium

area $= \frac{1}{2}(a+b)h$

Circle

circumference $= 2\pi r$

area $= \pi r^2$

Sector

area $= \frac{1}{2}r^2\theta$

arc length $= r\theta$

Cylinder

volume $= \pi r^2 h$

curved surface area $= 2\pi rh$

Cone

volume $= \frac{1}{3}\pi r^2 h$

curved surface area $= \pi rl$ where l = slant height

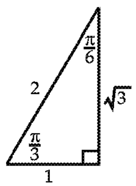
Sphere

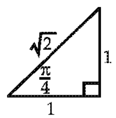
volume $= \frac{4}{3}\pi r^3$

surface area $= 4\pi r^2$

TRIGONOMETRY

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$





Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine Rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Identities

$$\cos^2\theta + \sin^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Compound Angles

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Double Angles

$$\sin 2A = 2\sin A\cos A$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

Products

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

Sums

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$
$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$
$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$
$$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

General Solutions

If $\sin\theta = \sin\alpha$ then $\theta = n\pi + (-1)^n\alpha$

If $\cos\theta = \cos\alpha$ then $\theta = 2n\pi \pm \alpha$

If $\tan\theta = \tan\alpha$ then $\theta = n\pi + \alpha$

where n is any integer

NUMERICAL METHODS

Trapezium Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h\left[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

where $h = \dfrac{b-a}{n}$ and $y_r = f(x_r)$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{1}{3}h\left[y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\right]$$

where $h = \dfrac{b-a}{n}$, $y_r = f(x_r)$ and n is even.