NZEST SCHOLARSHIP EXAMINATION 2003

MATHEMATICS WITH CALCULUS

TO ISSUE: STANDARD ANSWER BOOK

Friday 7 November 2003, 2pm

Time allowed: Three hours

Notes for Candidates

1 This examination consists of **SIX** questions, each worth 20 marks

2 Please attempt **ALL SIX** questions

3 You will get credit for every question (or part question) you do

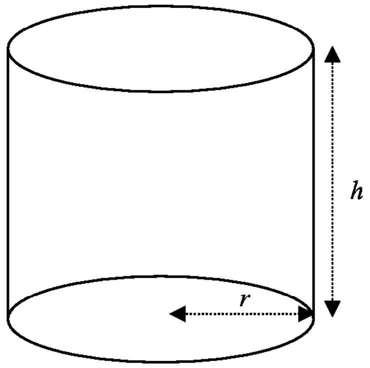
4 <u>Please show all working</u>

5 Calculators approved by NZQA may be used.

6 Tables of formulae are supplied as a detachable centrefold sheet.

Question 1 .

a. Part of the assembly for a biomedical pump consists of a cylinder, closed at both ends, which is made up of a very expensive composite material and is of constant thickness throughout. The aim is to obtain the maximum volume for a given mass of material. As the mass depends on the surface area, the problem is to find the dimensions that give the maximum volume for a given surface area.



Taking the radius of the cylinder to be to be r and the height to be h,
$A = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$ give the surface area and volume respectively.

(i) Obtain an expression for h in terms of A and r.

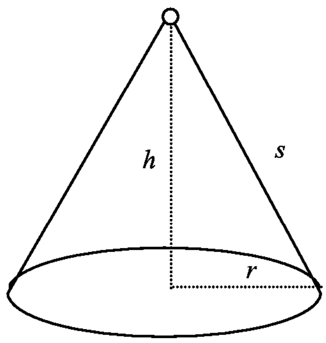
(ii) Show the volume can be written in the form $V = \frac{1}{2}Ar - \pi r^3$.

(iii) Find the value of r corresponding to the maximum volume.

(iv) Show this value of r does, in fact, give a maximum for V.

(v) By eliminating A between the formula for surface area and the expression for r found in **a.**(iii), or otherwise, find the corresponding value of h.

b. Another part of the assembly is a closed cone. As before, the problem is to find the dimensions that give the maximum volume for a given surface area.



Taking the radius of the cone to be to be r, the vertical height to be h and the slant height to be s,
$A = \pi r(s + r)$ and $V = \frac{1}{3}\pi r^2 h$ give the surface area and volume respectively.

(i) By finding expressions for $s + r$ and $s - r$, or otherwise, show

$$h = \frac{\sqrt{A(A - 2\pi r^2)}}{\pi r}$$

(ii) Find the value of r corresponding to the maximum volume. (You may assume that the volume is a maximum)

(iii) Find the corresponding values of s and h in terms of A.

Question 2 .

$\mathrm{H}_1 : xy = c^2$ and $\mathrm{H}_2 : x^2 - y^2 = a^2$ are two curves.

a. (i) What sort of curves are H_1 and H_2?

(ii) Sketch H_1 and H_2 on the same set of axes.
Give the co-ordinates of any intersections with the axes.
Show any asymptotes and give their equations.

(iii) By using implicit differentiation to find $\frac{dy}{dx}$ for each of the two curves, show that the tangents to each curve, at the point of intersection, meet at right angles.

b. (i) Show that H_1 can be represented by the parametric equations

$$x = ct, \quad y = \frac{c}{t}.$$

(ii) Find the equation of the normal to H_1 at the point T with co-ordinates $\left(ct, \frac{c}{t}\right)$.

Given that the normal to H_1 at the point T $\left(ct, \frac{c}{t}\right)$ meets H_1 again at P $\left(cp, \frac{c}{p}\right)$.

(iii) By finding the gradient of the line TP and comparing it with the gradient at T obtained by differentiating H_1, find a relationship between t and p.

(iv) Hence, or otherwise, by finding the parametric equations of M, the midpoint of TP, in terms of t show that the co-ordinates of M lie on the curve

$$4x^3y^3 + c^2\left(x^2 - y^2\right)^2 = 0$$

Note: This can also be stated as "Find the locus of M"

Question 3 .

a. Preliminary results:

(i) Find k if $\dfrac{A}{x(A-x)} = \dfrac{1}{x} + \dfrac{k}{A-x}$

(ii) Verify that $y = Ce^{kbx} - \dfrac{a}{b}$ is the solution to $\dfrac{dy}{dx} = k(a+by)$,
where C is an arbitrary constant and a, b are constants.

b. Rumours can spread quickly through a town. Assume that the population of the town is P and that only 1 person has heard the rumour initially. Here are two proposed models of the way rumours spread. The rate of increase in the number, n, of people who have heard the rumour is proportional to
Model A: The number that have not heard the rumour.
Model B: The product of the number that have heard the rumour and the number that have not.
Note : If y is proportional to x, then $y = kx$ for some constant k.

(i) The differential equation describing model A is $\dfrac{dn}{dt} = k(P-n)$ with the initial condition that $n = 1$ at $t = 0$.
Express model B as a differential equation.
(Note: The equation for Model A is given, and does not need deriving)

(ii) 1. Use the result in **a.**(ii) and substitute for a and b, or otherwise, to find the solution to Model A. Write your answer with the exponential term as the subject. Show your working.

2. Show, by using the result in **a.**(i) or otherwise, that the solution to the differential equation for Model B can be written as

$$e^{kPt} = \frac{n(P-1)}{P-n}.$$

(iii) Given $P = 20\,000$ and 20 people have heard the rumour in 5 days, the solution for Model A can be written as $e^{0.0002t} = \dfrac{19999}{20000-n}$.
Find the solution for model B, giving your answers in a similar form.
Express any decimals to 1 significant figure.

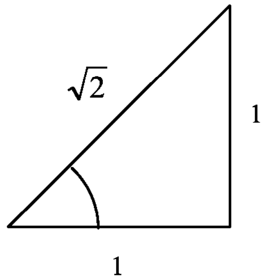
(iv) The smallest whole number of days that it takes for more than 99% of the population to have heard the rumour is 2300 days for model A.
What is the time for model B? Answer to the nearest day.

(v) What is the behaviour of both solutions as $t \to \infty$?

(vi) Peta claims model B is better than model A. Do you agree or not? Give a reason.

Question 4 .

a. (i) Use the following diagram to write down the exact values of $\sin 45^{\circ}$ and $\cos 45^{\circ}$.

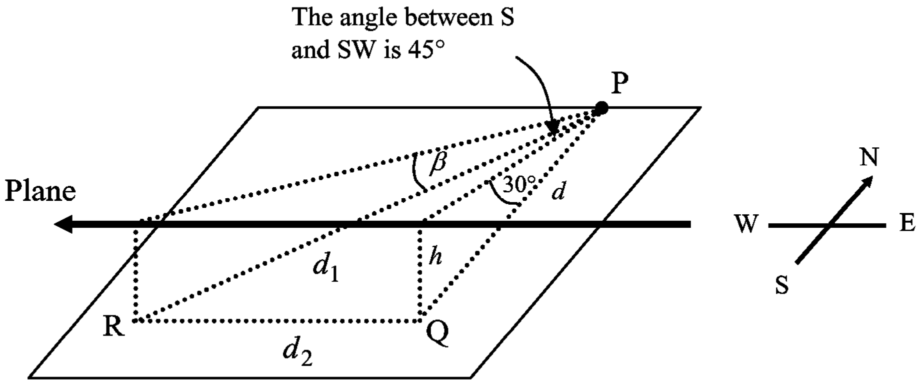


(ii) Prove that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

(iii) Hence, or otherwise, find an exact value for $\tan 22.5^{\circ}$.

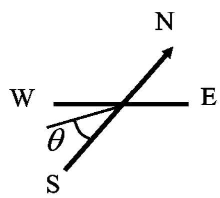
b. An aeroplane travels due west at a constant height. When it is at a point Q due south of a fixed point P its angle of elevation is 30°. We want to find its angle of elevation, β, when it is at a point R, south west of P.

The following diagram represents this situation. The aeroplane flies a distance d_2 at height h from a point immediately above Q to a point immediately above R. The distance from P to Q is d.



By forming 3 right-angled triangles or otherwise, find the elevation of the aeroplane, β, when it is immediately above R .

c. An aeroplane travels due west, climbing at an angle of elevation of φ. When it is due south of a fixed point P its angle of elevation is α. When it is at an angle θ south west of the given point its angle of elevation is β.



Show $\tan\varphi = \dfrac{\tan\beta - \cos\theta\tan\alpha}{\cos\theta\tan\theta}$

Question 5 .

Use radians and exact values throughout this question. In other words, do not use your calculator to evaluate any fractions or square roots and leave all angles as multiples of π.

a. (i) Given $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3}=\frac{1}{2}$, show $x^2-x+1=0$,

has roots $\cos\frac{\pi}{3}\pm i\sin\frac{\pi}{3}$.

(ii) Simplify $\frac{(cis2\theta)^5}{cis4\theta}$, giving the answer in terms of multiples of θ.

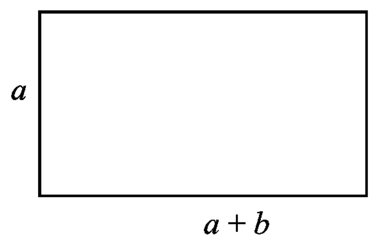
(iii) Show $cis(-\theta)=\cos\theta-i\sin\theta$.

Hence, or otherwise, assuming $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

show $(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$

b. The Golden Ratio, γ, occurs in many places in nature and in art. The diagram illustrates the golden rectangle with sides a and $a+b$,

where $\frac{a}{b}=\frac{b}{a+b}$.



$\gamma=\frac{a}{b}$ is called the Golden ratio.

It can be shown that $\frac{a}{b}=\frac{-1\pm\sqrt{5}}{2}$ Taking the positive root, $\gamma=\frac{-1+\sqrt{5}}{2}$.

Show the negative root can be written as $\frac{-1}{\gamma}$.

c. The Fibonacci sequence 1, 1, 2, 3, 5, ….. also occurs frequently in nature.
If we let u_n represent the nth term so that $u_1=1$, $u_2=1$, $u_3=2$...... it is defined by

$$\begin{cases} u_1=1 \\ u_2=1 \\ u_{n+2}=u_{n+1}+u_n \end{cases}.$$

For example, setting $n=1$ in the last expression gives $u_3=u_2+u_1=1+1=2$, $n=2$ gives $u_4=u_3+u_2=2+1=3$ and $n=3$ gives $u_5=u_4+u_3=3+2=5$.

(i) What are the next 2 terms of the sequence? (That is u_6 and u_7.)

The standard way to find an expression for u_n in terms of n, is to substitute $u_n = ax^n$ in the expression $u_{n+2} = u_{n+1} + u_n$ and so, for example, $u_{n+1} = ax^{n+1}$.

(ii) Substitute $u_n = ax^n$ in $u_{n+2} = u_{n+1} + u_n$, to obtain a quadratic equation which gives two possible values for x, α and β. Write these in terms of γ.

The most general solution for u_n can then be written as $u_n = A(-1)^n \gamma^n + \dfrac{B}{\gamma^n}$,

where A and B are constants.

(iii) Use your calculator to show $\gamma < 1$. Hence show that, as n gets large, the ratio $\left|\dfrac{u_n}{u_{n+1}}\right|$ approaches γ.

d. Consider the sequence u_n, defined by $\begin{cases} u_1 = 0 \\ u_2 = 1 \\ u_{n+2} = u_{n+1} - u_n \end{cases}$

(i) Write down the first 5 terms (that is u_1 to u_5).

The general solution is $u_n = A\alpha^n + B\beta^n$, where α, β are the roots of $x^2 - x + 1 = 0$.

(ii) Show that substituting $A = C - iD$, $B = C + iD$ this can be written as $u_n = C(\alpha^n + \beta^n) - iD(\alpha^n - \beta^n)$

(iii) Use **a.**(i) and (iii), to give an expression for u_n in terms of multiples of $\dfrac{\pi}{3}$.

(You will need to evaluate A and B.)

Question 6 .

a. Differentiate $\sin x\sqrt{1-9x^2}$

b. $u = u(x),\ v = v(x),\ w = w(x)$ are functions of x. Assuming the product rule for two functions, $\frac{duv}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$, by grouping uvw as $(uv)w$, prove the product rule for three functions: $\frac{d(uvw)}{dx} = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$

c. Two famous integrals are $F = \int\frac{dx}{\sqrt{1-k^2\sin^2 x}}$ and $E = \int\sqrt{1-k^2\sin^2 x}dx\ \ , k<1$

They are called elliptic integrals of the first and second kind respectively. (There is a third type we will not consider.) If an integral can be reduced to a function together with a combination of integrals of this type it is called an elliptic integral. It is known that elliptic integrals cannot be integrated exactly.

Defining Δ by $\Delta(x) = \sqrt{1-k^2\sin^2 x}$. we have $F = \int\frac{dx}{\Delta}$, $E = \int\Delta dx$.

The problem is to show that $I = \int\sqrt{1-k^2\sin^2 x}\cos^2 x dx$ is an elliptic function by showing it can be expressed in terms of elliptic integrals of the first and second kind.

(i) Find $\frac{d\Delta}{dx}$ expressing your answer in terms of $\sin x$, $\cos x$ in the numerator and Δ in the denominator.

(ii) Differentiate $\sin x\cos x\Delta$, bringing your answer to a common denominator of Δ.

(iii) By setting $v = \sin^2 x$ and using this to replace all occurrences of Δ^2, and $\cos^2 x$ as well, show that

$$\frac{d(\sin x\cos x\Delta)}{dx} = \frac{1}{\Delta}\left[1 - 2(k^2+1)v + 3k^2v^2\right]$$

Hence $\sin x\cos x\Delta = \int\frac{dx}{\Delta} - 2(k^2+1)\int\frac{vdx}{\Delta} + 3k^2\int\frac{v^2dx}{\Delta}$

(iv) Show that $I = \frac{1}{3}\Delta\sin x\cos x + \frac{1+k^2}{3k^2}E - \frac{1-k^2}{3k^2}F$