S

93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2010
Mathematics with Calculus

9.30 am Saturday 27 November 2010
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

Pull out the Formulae and Tables Booklet S–CALCF from the centre of this booklet.

There are FIVE questions in this booklet. Answer ALL questions.

Write your answers in the Answer Booklet 93202A.

Show all working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You have three hours to complete this examination.

QUESTION ONE

(a) Show that $z = \text{cis}\dfrac{7\pi}{11}$ is a solution of $z^{19} + z^{14} + z^8 + z^3 = 0$.

You should **not** attempt to find any other solutions of this polynomial.

(b) Find all **real** values of x which satisfy the following set of equations.

$$\sin x = \sqrt{y}$$

$$12y^3 - 13y^2 - 14y + 13 = \cos^2 x$$

Start by finding possible values of y.

(c) Find the complex roots of $\dfrac{z^{12} - 1}{(z^4 - 1)(z^3 - 1)} = 0$.

You may find it helpful to work in **polar form**.

QUESTION TWO

(a) Figure 1 below shows four hyperbolae, where $\alpha = 1 + \sqrt{2}$.

$$x^2 - y^2 = 2\alpha$$
$$y^2 - x^2 = 2\alpha$$
$$xy = \alpha$$
$$xy = -\alpha$$

You may make use of the eight-fold symmetries of the hyperbolae in your answers to this question.

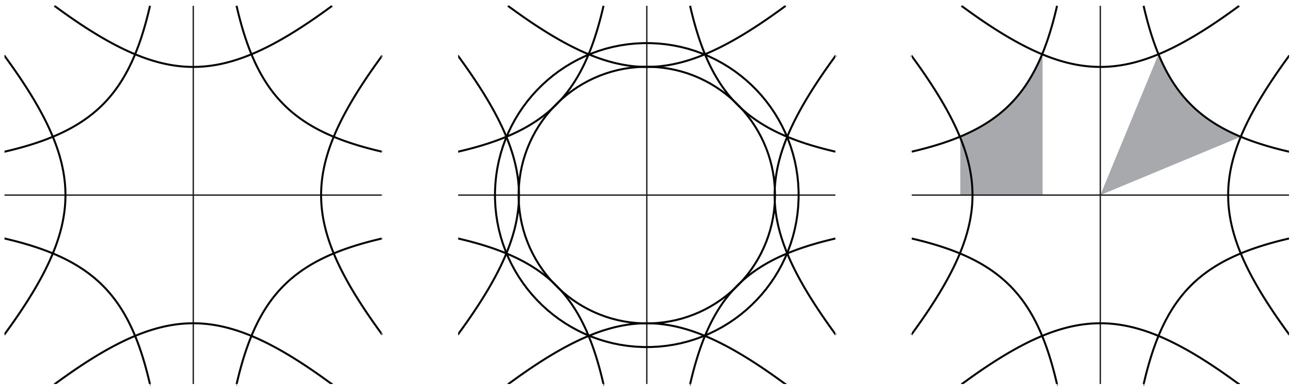

Figure 1: Intersecting hyperbolae. Figure 2: Inscribed circle and circumscribed circle. Figure 3: Two shaded regions of equal area.

(i) Figure 2 shows one circle inscribed in the central region and another circle circumscribed around it.

Find the **exact** area of each circle.

(ii) Note that the areas of the two shaded regions in Figure 3 are the same. The area of the entire inner region in Figure 1 is eight times the area of a shaded region in Figure 3.

Use integration to find the **exact** area of the entire inner region.

(b) Recall that the points of inflection of a curve are places where the second derivative changes sign. These are typically, **but not always**, points at which the second derivative is zero.

Consider the curve $y = \sqrt[3]{x} \cdot e^{-x^2}$.

Write the second derivative in the form $\dfrac{d^2y}{dx^2} = (ax^4 + bx^2 + c)e^{-x^2}x^{-5/3}$, and hence find the x-coordinates of the points of inflection of the curve.

QUESTION THREE

(a) A flower pattern is constructed by using a sinusoidal function $r(\theta)$ to define the distance from the origin to the curve at a radial angle θ (measured anti-clockwise from the positive x-axis).

The x and y coordinates of a point on the curve are given by the following equations, where $0 \le b \le a$ and n is a positive integer (the number of petals).

$r(\theta) = a + b\sin(n\theta)$ $\qquad x(\theta) = r(\theta)\cdot\cos(\theta)$ $\qquad y(\theta) = r(\theta)\cdot\sin(\theta)$

Three flower patterns with seven petals are shown in Figure 5 below.

(i) The area inside such a function is given by $A = \frac{1}{2}\int_0^{2\pi} \left(r(\theta)\right)^2 \mathrm{d}\theta$ when $r(\theta) \ge 0$.

Show that the area of a flower pattern is $\pi\left(a^2 + \frac{1}{2}b^2\right)$.

(ii) A lemon squeezer (see Figure 4) with base radius a_0 and height H is made to the following specifications.

At a height h (where $0 \le h \le H$) the cross-section is a flower pattern with

$$a(h) = \frac{H-h}{H}a_0 \text{ and } b(h) = \frac{h}{H}\cdot a(h).$$

Use integration with respect to h to show that the volume of the lemon squeezer is **exactly 5% greater than** the volume of a cone with the same base and height.

Figure 4: A lemon squeezer of vertical height H.

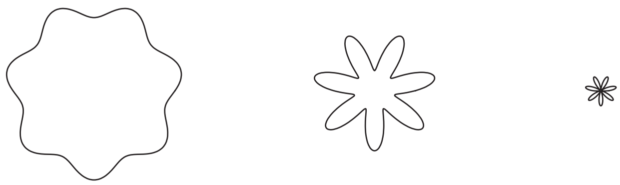

Figure 5: Three flower patterns which are cross-sections of the lemon squeezer at $h = 0.1H$, $h = 0.5H$ and $h = 0.9H$ respectively.

(b) Find the **ranges** of values of α in the interval $0 \le \alpha \le \pi$ for which the roots of the following quadratic equation are real, where p is a constant.

$$x^2\sin^2\alpha + px\left(\sqrt{3}\sin\alpha + \cos\alpha\right) + p^2 = 0$$

Start by showing that, when the roots are real, $\sqrt{3}\sin(2\alpha) + \cos(2\alpha) \ge 0$.

Give your answer in **exact** form.

QUESTION FOUR

(a) Figure 6 shows the parabolae $y = (x - a)^2$, $y = (x + a)^2$ and $y = bx^2 - k$, where $0 < b < 1$.

The third parabola meets each of the others at a point with a common tangent line.

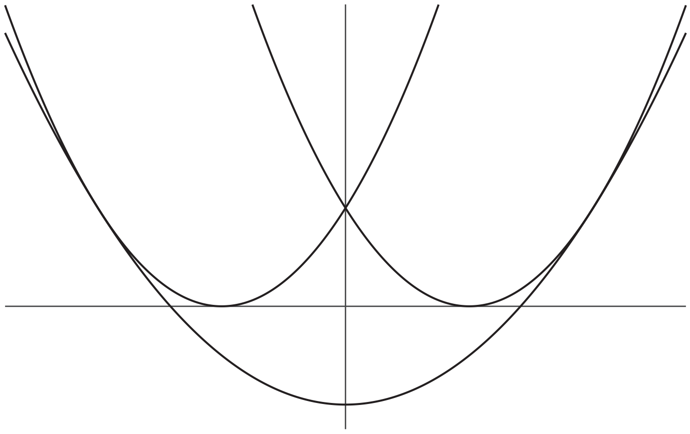

Figure 6: Parabolae $y = (x - a)^2$, $y = (x + a)^2$ and $y = bx^2 - k$.

Find k in terms of a and b.

(b) Prove that the length ℓ of a tangent line from an external point (p,q) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $\ell = \sqrt{p^2 + q^2 + 2gp + 2fq + c}$.

Draw a diagram to support your answer.

(c) The difference between the x-coordinates of two points on the parabola $y^2 = 4ax$ is fixed at $2k$, where k is a constant, as shown in Figure 7.

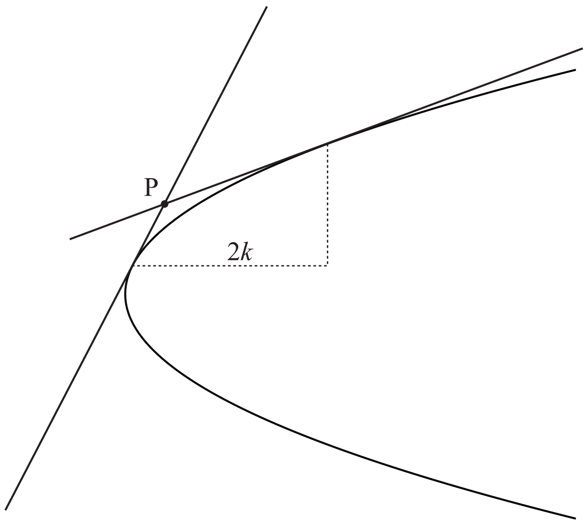

Figure 7: Intersection of tangents to $y^2 = 4ax$.

Find an equation that describes the position (x_P, y_P) of the point of intersection P of the tangents at the two points.

Write your answer in the form $y_P^2 = f(x_P)$, and hence show that $4ax_P < y_P^2 < 4ax_P + 2ak$.

QUESTION FIVE

Figure 8 below shows a discussion about how the size of the dating pool changes as a person ages.

For copyright reasons, this resource cannot be reproduced here.

Figure 8: A comic strip adapted from the xkcd web-comic at www.xkcd.com/314/

A simple model fitting the general shape shown in the comic strip for the proportion of singles of age t is $S(t) = \mathrm{e}^{-0.05t}$ for $t \geq 18$ (where t is measured in years).

Suppose, for the sake of this question, that the model is accurate, and that all singles date according to the rule given in the comic strip.

(a) Find a rule for the upper limit for the dateable range of a person of age T, and hence the width of the dateable range for a person of age T.

(b) Integrate $S(t)$ between the lower and upper limits to find a measure of the dating pool for a person of age T.

(c) Find the age at which the dating pool is largest, correct to one decimal place.

(You need not show that your answer is a maximum.)

93202Q