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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2011
Mathematics with Calculus

9.30 am Saturday 26 November 2011
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are FIVE questions in this booklet. Answer ALL questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–7 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You have three hours to complete this examination.

QUESTION ONE (8 marks)

(a) Find all **real** solutions of $\dfrac{ae^x}{2e^x - 1} < 1$, where a is a positive constant.

(b) A three-dimensional solid has a surface made of two types of triangles, types A and B.

The area of a Type A triangle is $A_{\mathrm{A}} = \sqrt{2k^2 - 2k + 1}$.

The area of a Type B triangle is $A_{\mathrm{B}} = \dfrac{\sqrt{3}}{2}\left(k^2 - k + 1\right)$.

Find the value of k which minimises the total surface area, and show that this value gives a minimum.

(c) Find **all** functions $y = f(x)$ which satisfy $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^{m+1}$, where m is a non-zero constant.

Show that these functions also satisfy $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(y^n\right) = ny^{n+m}$.

QUESTION TWO (8 marks)

(a) A circular pond is 5 metres in radius. The volume (in cubic **metres**) of water in the pond when the water is x metres from the top is $V(x) = \dfrac{250\pi}{3} - 25\pi x + \dfrac{\pi}{3}x^3$.

Rain falls at the rate of 15 **millimetres** per hour.

How fast is the depth of water in the pond rising when it is 3 metres from the top?

(b) Figure 1 below shows the function $g(x) = \sqrt{1-\sqrt{x}}$ as a solid line.

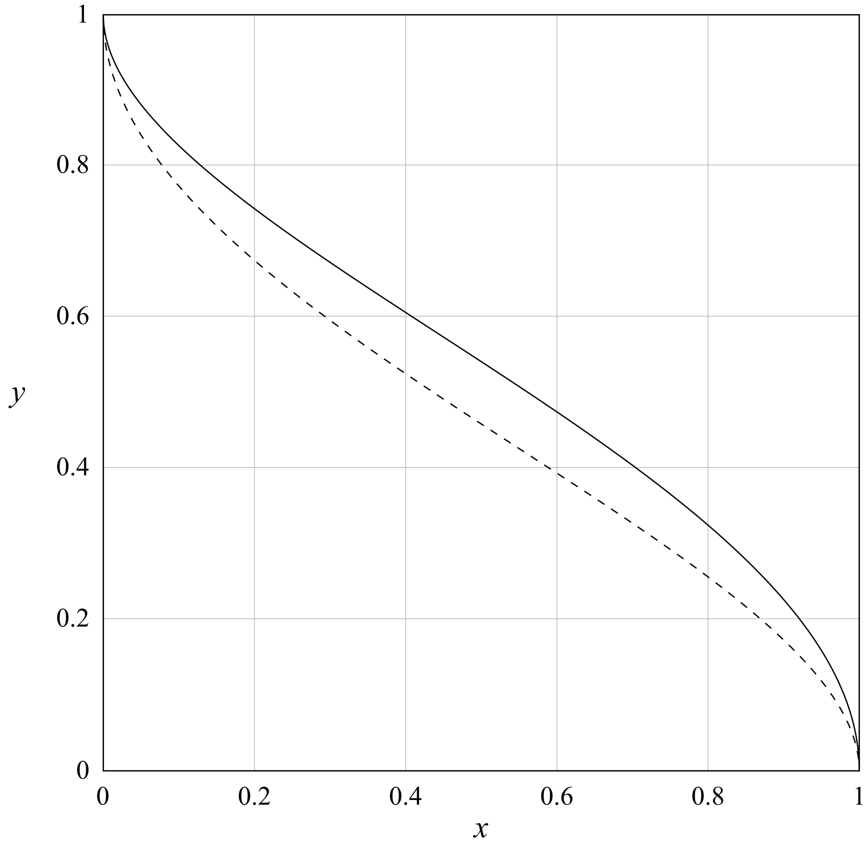

Figure 1: Graph of $y = g(x)$, and the curve rotated about the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

(i) Use differentiation to show that $\displaystyle\int g(x)\,\mathrm{d}x = A\left(1-\sqrt{x}\right)^{1.5}\left(2+3\sqrt{x}\right)+C$, and find the value of A.

(ii) Figure 1 shows the curve $y = g(x)$ rotated 180 degrees around the point $\left(\frac{1}{2},\frac{1}{2}\right)$ as a dotted line. The two curves intersect only at the points (0,1) and (1,0).

Find the area of the region enclosed by the two curves.

QUESTION THREE (8 marks)

(a) Make use of trigonometric identities to find the **exact** value of $\cos\left(\frac{7\pi}{12}\right)$.

(b) Two **half-angle** formulae for trigonometry are given below.

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Given that $\tan\theta = 20\sqrt{6}$ and $0 < \theta < \frac{\pi}{2}$, find an exact value of $\tan\left(\frac{\theta}{4}\right)$.

Simplify your answer.

(c) The Moeraki boulders are natural stone spheres sunk into the sand of Moeraki Beach between Oamaru and Dunedin. Figure 2 shows some Moeraki boulders partially submerged.

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Figure 2: Spherical boulders near Moeraki.
http://tinyurl.com/3jqt458

The angle between the surface of the water and a tangent plane to the boulder is ϕ, as shown in Figure 3.

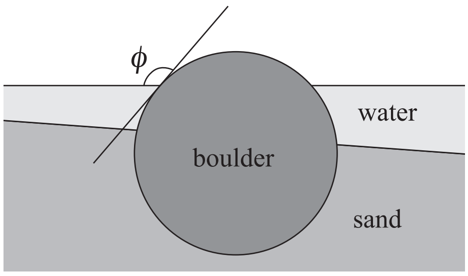

Figure 3: A cross-section of the side view of a partially submerged Moeraki boulder.

Find the **proportion** of the volume of the boulder which is below water level.

QUESTION FOUR (8 marks)

(a) Two circles are defined by the following equations where $r < a$.

$$(x - a)^2 + y^2 = r^2$$
$$(x + a)^2 + y^2 = r^2$$

Find the equations of the **four** lines which are tangent to **both** circles.

(b) An elliptical frame is built with two equally bright light bulbs at the focal points of the ellipse, as shown in Figure 4.

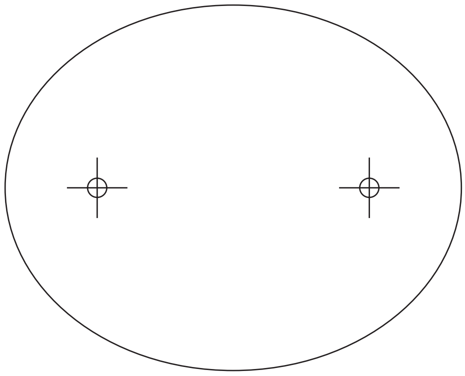

Figure 4: Elliptical frame with bulbs at focal points.

At any point on the frame, the sum of the distances to the bulbs is a constant d. The brightness of a bulb at the frame is inversely proportional to the square of the distance to the bulb.

An expression for the brightness at a point on the frame distance x from the left bulb is

$B(x) = I\left(\frac{1}{x^2} + \frac{1}{(d - x)^2}\right)$, where I and d are constants.

Find the least brightly lit points on the frame, giving reasons for your answers.

(c) A gunshot is heard first by one observer and then 1.5 seconds later by a second observer three kilometres away. Assume that the gun and the ear levels of the observers are all in the same horizontal plane and that the speed of sound is 340 metres per second.

Choosing an appropriate coordinate system, write an equation that describes the possible positions of the gun, relative to the observers.

Draw a diagram to illustrate your answer, indicating the possible positions of the gun relative to the two observers.

QUESTION FIVE (8 marks)

(a) The **gamma function** Γ extends the factorial function to all of the real numbers except for the negative integers and zero.

It takes values of the factorial function at positive integers: $\Gamma(n + 1) = n!$ and has the property that for any values, $\Gamma(x) = (x - 1)\Gamma(x - 1)$.

A formula useful for finding other values of the gamma function is Euler's reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Find the **exact** value of $\Gamma\left(\frac{1}{2}\right)$, and hence the exact value of $\Gamma\left(\frac{5}{2}\right)$.

(b) Consider the equation $x^6 + (2 - k)x^4 + (25 - 2k)x^2 - 25k = 0$, where k is a **real** constant.

Given that $c = \sqrt{2} + \sqrt{3}\,\mathrm{i}$ is a root of this equation, find the other five roots.

(c) The vertices of a regular octagon, shown on the Argand diagram in Figure 5, represent the roots of a complex degree 8 polynomial. One root is shown at the point A, 1 + 0i.

Write the polynomial in the form $p(z) = (z - (a + b\mathrm{i}))^n + q$.

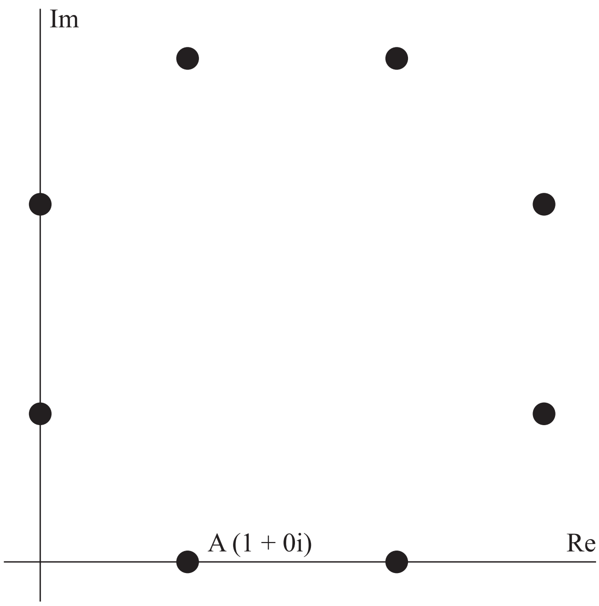

Figure 5: Roots of a polynomial on an Argand diagram, including the point A, 1 + 0i.