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93202Q

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NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2012
Mathematics with Calculus

9.30 am Saturday 24 November 2012
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are FIVE questions in this booklet. Answer ALL questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Start your answer to each question on a new page. Carefully number each question.

Check that this booklet has pages 2–7 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You have three hours to complete this examination.

QUESTION ONE (8 marks)

(a) (i) Show that $\left(\sqrt[3]{a+b}+\sqrt[3]{a-b}\right)^3 = 2a+3\cdot\sqrt[3]{a^2-b^2}\left(\sqrt[3]{a-b}+\sqrt[3]{a+b}\right)$.

(ii) The equation $x^3 = x + 1$ has a unique real solution.

Show that this solution is $\rho = \sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}$.

(b) Stewie Griffin is a character from the television programme *Family Guy*.

His head can be considered as a volume of revolution, turning a curve on an axis passing through his ears, as shown in Figure 1.

Different volumes are obtained, depending on the shape of the rotated curve.

Assuming Stewie's head has width $2w$ and height $2h$, find the **ratio** of the volume obtained using a parabolic curve to the volume obtained using a semi-elliptical curve.

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Figure 1: Stewie Griffin's head represented as a volume of revolution.

http://freeimagesarchive.com/data/media/48/stewie+griffin.gif

QUESTION TWO (8 marks)

(a) Consider the equation $x^n = \tan(ny)$, where n is a constant.

Find an expression for $\frac{dy}{dx}$ as a function of x.

(b) Consider the family of ellipses given below, where $r \neq 0$, and B and C are fixed constants.

$$Bx^2 + Cy^2 = r^2$$

Find an expression for $\frac{dy}{dx}$, and hence find an equation in the form $y = f(x)$ for the family of curves that always intersect these ellipses at right angles.

(c) Models of exponential growth and decay arise in many diverse fields, eg:

- radioactive decay
- bacterial growth
- epidemic spread
- drug metabolism
- compound interest.

Use calculus to explain why the simple equation $A(t) = A_0 e^{kt}$ models such a wide range of real world situations.

Write at most half a page.

QUESTION THREE (8 marks)

(a) (i) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(x\cos(x)\big)$ and use this result to find $\int x\sin(x)\,\mathrm{d}x$.

(ii) Hence find the value of $\int\limits_0^{n\pi} x\sin(x)\,\mathrm{d}x$ for positive integer values of n.

(b) Consider the points in the region R shown in the Argand diagram of Figure 2, consisting of all points in a right-angled sector of radius 1, except for the point $z = 0.8\,\mathrm{cis}\,\dfrac{\pi}{6}$.

Sketch the region containing all points w^3, where w is a point within the region R.

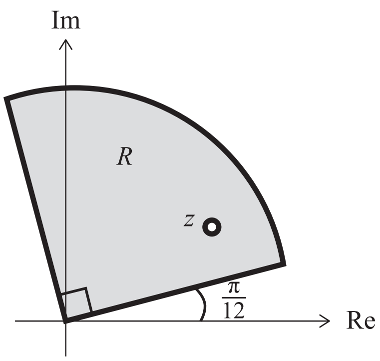

Figure 2: The region R in an Argand diagram (note the point z is not a point in R).

QUESTION FOUR (8 marks)

(a) Consider the function $f(x) = \log_m x + \log_x m$ defined for $x > 1$.

For a fixed value of $m > 1$, find the minimum value of $f(x)$.

Clearly explain the steps of your working.

(b) The following equation relates two real variables x and y, where q is a fixed constant.

$$y^4 + (1 - q^2)x^2y^2 - q^2x^4 + q^2x^2 - y^2 = 0$$

Sketch all points that satisfy the equation.

You might start by substituting $y^2 = q^2x^2$ and interpreting the result.

(c) Prove the following trigonometric identity:

$$2\tan(2x) \cdot \left(\tan(x) - 1\right)^2 \left(\tan(x) + 1\right)^2 = \tan(4x) \cdot \left(\tan^2(x) - 2\tan(x) - 1\right)\left(\tan^2(x) + 2\tan(x) - 1\right)$$

Note that you do not need to work in terms of $\sin(x)$ and $\cos(x)$ to prove this identity.

QUESTION FIVE (8 marks)

(a) Figure 3 shows six lines passing through the origin. The lines are separated by equal angles. Some exact values of tan(t) are given in Table 1.

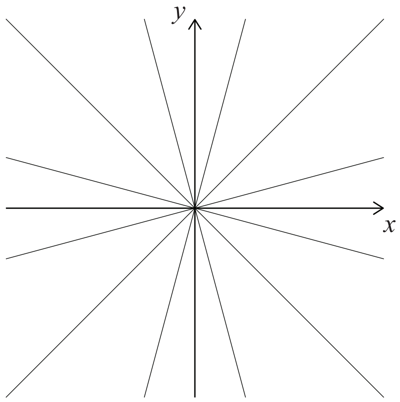

Figure 3: Six lines through the origin separated by equal angles.

t	$\tan t$
$\frac{1}{12}\pi$	$2-\sqrt{3}$
$\frac{1}{6}\pi$	$\sqrt{\frac{1}{3}}$
$\frac{1}{4}\pi$	1
$\frac{1}{3}\pi$	$\sqrt{3}$
$\frac{5}{12}\pi$	$2+\sqrt{3}$

Table 1: Selected exact values of $\tan t$.

(i) Show that the lines can be represented by the following equation:

$$\left(x^2-y^2\right)\left(x^2-\left(7-4\sqrt{3}\right)y^2\right)\left(x^2-\left(7+4\sqrt{3}\right)y^2\right)=0$$

(ii) Find an equation for a hyperbola that does not cross **any** of the six lines in Figure 3, giving reasons for your answer.

(b) A collection of parallel chords connect pairs of points on an ellipse, as shown in Figure 4.

For the general ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the midpoints of the chords lie on a straight line.

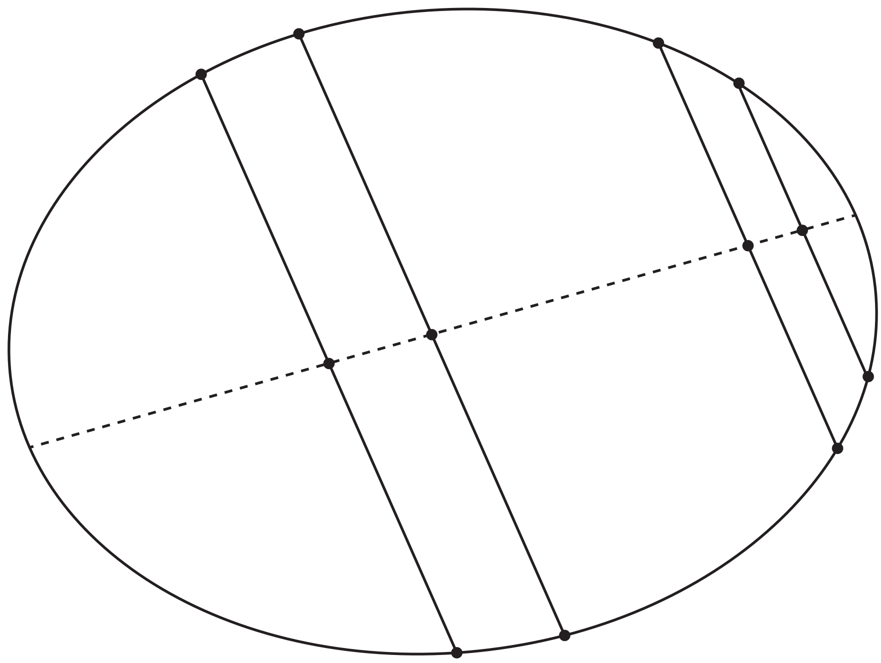

Figure 4: Parallel chords of an ellipse with their midpoints on a line.